A STOCHASTIC INVESTMENT MODEL FOR SOUTH AFRICAN USE

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There continues to be a global need for stochastic modelling or “Economic Scenario Generators”

Models for both the movement of assets and liabilities

Required by regulation in many instances

The first published stochastic investment model was by Wilkie in 1986 (updated and extended)
Question

Do I need to learn all of the long formulae for the Wilkie model and the term structure models?

Answer

We don’t think it will be necessary to memorise all of these formulae, but it is important that you know some of the key formulae, such as Wilkie’s inflation equation and the stochastic differential equations defining each term structure model.

In past 109 and CT8 papers, any detailed equations that you needed were given in the question. You were then asked to manipulate them and/or explain their main features.
Limited research has been done in South Africa

The seminal paper was done by Thomson in 1996 (a model which by his own account had a number of practical limitations)
1. Introduction

  - first comprehensive stochastic model for actuarial applications
  - cascade structure – price inflation is the driving force
  - yearly data
  - Data from 1919-1982 (63 years of data)
  - long-term forecasting

  - SA data (price inflation, rental yields and growth rates, short-term and long-term interest rates, dividend yields and growth rates)
  - Wilkie type model with a cascade structure too (different)
  - Data from 1960-1993
  - Thomson & Gott developed a long term Equilibrium model (different– no arbitrage)
1. Introduction

- The aim of the paper is to propose an open source, “real world” stochastic investment model for long-term applications for SA use.

- We propose a stochastic investment model for South African use by modelling:
  - Price inflation;
  - Short and long term interest rates;
  - Inflation-linked bond yields; and
  - Local equity returns.


- We use data from 1960-2018 (except for ILBs).

- Main contribution is therefore development of SA model based on longer data set and incorporating inflation-linked yields.
2. Data

- Data from 1960-2018 has been used

- **Inflation Data**
  - South African Consumer Price Index (CPI) provided by Statistics South Africa

- **Equity Data**
  - Indices used are JSE-Actuaries All Share Index (CI101) by INET and ADY (Div Yield)
  - Data updated until 2001 by IRESS but data then used is J203 (ALSI Total Return Index) and J202 (ALSI Dividend Yield)
2. Data

- **Long-term interest bearing securities**
  - JSE-Actuaries Long Bond Yield (i.e. the 20 year Bond Yield) is used
  - Longer than the average outstanding term of quoted long-term interest-bearing securities
  - Code is JAYC20 (Previously data provided by INET and updated by IRESS)

- **Short-term interest rates**
  - Thomson originally used data provided by Ginsburg Malan and Carsons Money- Market Index (GMC1)
  - Now referred to as the Alexander Forbes Money-Market Index and was provided by IRESS.

- **Index-linked bonds**
  - The yields on SA Government bonds in issue since first issuance in March 2000 used
  - Data provided by Bloomberg
3. Structure of the Model

- Index-linked bonds
  - The yields on SA Government bonds in issue since first issuance in March 2000 used
    - Data provided by Bloomberg
4. Price Inflation

Price Inflation

Annual force of inflation based on CPI, 1960-2018
4. Price Inflation

- Model the force of inflation as a stationary autoregressive process (AR(1) model)
- Force of inflation is the difference between the logarithms of CPI each year
- Inflation-targeting introduced in 2000 (target band of 3%-6%)
- In long run, the mean and variance of an AR(1) model are constant
The model for the Consumer Price Index (CPI), where $Q_t$ is the value of CPI at time $t$, is:

\[
\begin{align*}
Q_t &= Q_{t-1}. \exp(\delta_q(t)) \\
\delta_q(t) &= \mu_q + a_q.(\delta_q(t-1) - \mu_q) + \epsilon_q(t) \\
\epsilon_q(t) &= \sigma_q.z_q(t) \\
\text{iid} & \quad \sim N(0, 1)
\end{align*}
\]

where $\mu_q$ is the long-run mean, $a_q$ is the autoregressive parameter, $\sigma_q$ is the standard deviation of the residuals and $z_q$ is unit normal variables.
4. Price Inflation

<table>
<thead>
<tr>
<th>$\delta_q(t)$</th>
<th>1960-2018</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_q$</td>
<td>0.0809 (0.0185)</td>
</tr>
<tr>
<td>$\alpha_q$</td>
<td>0.8433 (0.0670)</td>
</tr>
<tr>
<td>$\sigma_q$</td>
<td>0.0220 (0.0020)</td>
</tr>
</tbody>
</table>

Log Likelihood: 139.05

$r_z(1)$: -0.119
$r_{z^2}(1)$: -0.043

Skewness $\sqrt{\beta_1}$: -0.1031
Kurtosis $\beta_2$: 2.7841
Jarque-Bera $\chi^2$: 0.2191
$p(\chi^2)$: 0.8962

- All parameters are significant
- Residuals are independent and normally distributed
- No ARCH effect
- Long-term mean will be lowered for practical use
- Skewness and Kurtosis coefficients close to Normal Disn
5. Parameter Stability

- Analysis for long term stability key as parameters assumed to be constant for forecast period

  - Recursively estimate parameters on incrementally larger data sets
  - Solid lines show parameter estimates and dotted lines are 95% CIs
  - Heavy lines start in 1960 and end in given year (min 25 obs)
  - Thinner lines show periods ending in 2018 (min of 10 obs)
Estimates for parameter $\mu_q$
5. Parameter Stability

Estimates for parameter $\sigma_q$
6. Share Dividend Yields

Dividend Yields %, 1961-2018
6. Share Dividend Yields

- We fitted several models to dividend yields
- Final model included both autoregressive and inflation effects

$y(t)$ is the yield on the index at time $t$, $ym(t)$ is the moving average effect of inflation on dividend yields.

\[
\begin{align*}
ym(t) &= d_y \cdot \delta_q(t) + (1 - d_y) \cdot ym(t - 1) \\
y_q(t) &= w_y \cdot ym(t) + (1 - w_y) \cdot \delta_q(t) \\
\ln y(t) &= y_q(t) + \mu_y + ym(t) \\
y_n(t) &= a_y \cdot y_n(t - 1) + \epsilon_y(t) \\
\epsilon_y(t) &= \sigma_y \cdot z_y(t) \\
z_y(t) &\overset{iid}{\sim} N(0, 1)
\end{align*}
\]

that is $z(t)$ is a series of independent, identically distributed unit normal variates.
### 6. Share Dividend Yields

<table>
<thead>
<tr>
<th></th>
<th>1962-2018</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_y(t)$</td>
<td></td>
</tr>
<tr>
<td>$w_y$</td>
<td>-4.0074  (1.2161)</td>
</tr>
<tr>
<td>$d_y$</td>
<td>0.1396  (0.0557)</td>
</tr>
<tr>
<td>$\mu_y$</td>
<td>1.2695  (0.1774)</td>
</tr>
<tr>
<td>$a_y$</td>
<td>0.8266  (0.0727)</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>0.2261  (0.0214)</td>
</tr>
<tr>
<td>Log Likelihood</td>
<td>3.81</td>
</tr>
<tr>
<td>$r_z(1)$</td>
<td>-0.049</td>
</tr>
<tr>
<td>$r_z^2(1)$</td>
<td>0.180</td>
</tr>
<tr>
<td>skewness $\sqrt{\beta_1}$</td>
<td>0.5254</td>
</tr>
<tr>
<td>kurtosis $\beta_2$</td>
<td>4.0364</td>
</tr>
<tr>
<td>Jarque-Bera $\chi^2$</td>
<td>5.1731</td>
</tr>
<tr>
<td>$p(\chi^2)$</td>
<td>0.0753</td>
</tr>
</tbody>
</table>

- All parameters are significant
- Significant cross correlation between the price inflation and dividend yields
- Weighted moving average effect of inflation improves the model significantly
- Residuals are independent and normally distributed
- No ARCH effect
7. Share Dividends

![Graph showing the trend of share dividends from 1960 to 2020.](image-url)
7. Share Dividends

\[ D_t = P_t \cdot Y_t \]

where \( D_t \) is share dividends, \( P_t \) is share prices and \( Y_t \) is share dividend yields. The logarithm of the dividend growth,

\[ \delta_d(t) = \ln D_t - \ln D_{t-1} \]

\[
\begin{align*}
    dm(t) &= d_d \cdot \delta_q(t) + (1 - d_d) \cdot dm(t - 1) \\
    d_q(t) &= w_d \cdot dm(t) + (1 - w_d) \cdot \delta_q(t) \\
    \delta_d(t) &= d_q(t) + \mu_d + y_d \cdot \epsilon_y(t - 1) + k_d \cdot \epsilon_d(t - 1) + \epsilon_d(t) \\
    \epsilon_d(t) &= \sigma_d \cdot z_d(t) \\
    z_d(t) &\overset{\text{iid}}{\sim} N(0, 1)
\end{align*}
\]
7. Share Dividends

<table>
<thead>
<tr>
<th>1962-2018</th>
<th>( \delta_d(t) )</th>
<th>only ( \delta_y ) effect</th>
<th>+ Inflation effect</th>
<th>+ MA Inflation effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_d )</td>
<td></td>
<td>-4.7651 (2.9754)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_d )</td>
<td></td>
<td>0.6164 (0.2052)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( q_d )</td>
<td></td>
<td>0.8462 (0.4902)</td>
<td>0.0885 (0.04690)</td>
<td>0.0784 (0.0255)</td>
</tr>
<tr>
<td>( \mu_d )</td>
<td>0.1549 (0.0278)</td>
<td>0.0885 (0.04690)</td>
<td>0.0784 (0.0255)</td>
<td></td>
</tr>
<tr>
<td>( y_d )</td>
<td>-0.1739 (0.0742)</td>
<td>-0.1773 (0.0712)</td>
<td>-0.1975 (0.0698)</td>
<td></td>
</tr>
<tr>
<td>( k_d )</td>
<td>0.3438 (0.1172)</td>
<td>0.3660 (0.1141)</td>
<td>0.3260 (0.01217)</td>
<td></td>
</tr>
<tr>
<td>( \sigma_d )</td>
<td>0.1222 (0.0115)</td>
<td>0.1190 (0.0112)</td>
<td>0.1116 (0.0105)</td>
<td></td>
</tr>
<tr>
<td>Log Likelihood</td>
<td>38.27</td>
<td>39.76</td>
<td>43.34</td>
<td></td>
</tr>
<tr>
<td>( r_z(1) )</td>
<td>0.009</td>
<td>0.031</td>
<td>0.038</td>
<td></td>
</tr>
<tr>
<td>( r_{z2}(1) )</td>
<td>0.179</td>
<td>0.181</td>
<td>0.137</td>
<td></td>
</tr>
<tr>
<td>skewness ( \sqrt{\beta_1} )</td>
<td>0.0187</td>
<td>-0.0293</td>
<td>0.0051</td>
<td></td>
</tr>
<tr>
<td>kurtosis ( \beta_2 )</td>
<td>3.9069</td>
<td>3.6078</td>
<td>3.1577</td>
<td></td>
</tr>
<tr>
<td>Jarque-Bera ( \chi^2 )</td>
<td>1.9566</td>
<td>0.8855</td>
<td>0.0593</td>
<td></td>
</tr>
<tr>
<td>( p(\chi^2) )</td>
<td>0.3759</td>
<td>0.6423</td>
<td>0.9708</td>
<td></td>
</tr>
</tbody>
</table>

- 6 parameters to estimate
- All models fit the data well
- Model with both dividend yield and moving average inflation effect seem best
- Residuals are independent and normally distributed
- No ARCH effect
8. Long-term Interest Rates
8. Long-term Interest Rates

For long-term interest rates $\delta_c(t)$, the JSE-Actuaries Long Bond Yield (i.e. JAYC20, the 20-year Bond Yield) is used as in Thomson (1996).

\[
\begin{align*}
  cm(t) &= d_c \cdot \delta_q(t) + (1 - d_c) \cdot cm(t - 1) \\
  cr(t) &= \delta_c(t) - w_c \cdot cm(t) \\
  ln \ cr(t) &= ln \ \mu_c + cn(t) \\
  cn(t) &= a_c \cdot cn(t - 1) + \epsilon_c(t) \\
  \epsilon_c(t) &= \sigma_c \cdot z_c(t) \\
  z_c(t) &\sim iid \ N(0, 1)
\end{align*}
\]
8. Long-term Interest Rates

<table>
<thead>
<tr>
<th>Parameter</th>
<th>1961-2018</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta_c(t)$</td>
<td>1961-2018</td>
</tr>
<tr>
<td>$w_c$</td>
<td>1.0</td>
</tr>
<tr>
<td>$d_c$</td>
<td>0.13</td>
</tr>
<tr>
<td>$\ln \mu_c$</td>
<td>-3.3892 (3.37%) (0.1086)</td>
</tr>
<tr>
<td>$a_c$</td>
<td>0.5665 (0.1117)</td>
</tr>
<tr>
<td>$\sigma_c$</td>
<td>0.3610 (0.0341)</td>
</tr>
<tr>
<td>$r_z(1)$</td>
<td>-0.108</td>
</tr>
<tr>
<td>$r_z^2(1)$</td>
<td>0.109</td>
</tr>
<tr>
<td>skewness $\sqrt{\beta_1}$</td>
<td>-0.9368</td>
</tr>
<tr>
<td>kurtosis $\beta_2$</td>
<td>4.2522</td>
</tr>
<tr>
<td>Jarque-Bera $\chi^2$</td>
<td>12.06</td>
</tr>
<tr>
<td>$p(\chi^2)$</td>
<td>0.0024</td>
</tr>
</tbody>
</table>

- **Fisher relation:**
  Nominal rates = Real interest + Implied Inflation

- Exponentially weighted moving average effect of inflation
- All parameters are significant
- Residuals are independent but not normally distributed
- No ARCH effect
As in Thomson (1996) the Ginsburg Malan & Carsons Money-Market Index has been used which is under the code $GMC1$. The short-term interest rates, $\delta_b(t)$, is obtained as

$$
\delta_b(t) = \ln \frac{GMC1_t}{GMC1_{t-1}}
$$

$$
\ln \delta_c(t) - \ln \delta_b(t) = - \ln(\delta_b(t)/\delta_c(t))
$$
i.e. the logarithm of the ratio of the rates.
9. Short-term Interest Rates

![Graph showing short-term interest rates over time with peaks and troughs from 1970 to 2020.](image-url)
9. Short-term Interest Rates

\[ \delta_b(t) = \delta_c(t) \cdot \exp(-bd(t)) \]

\begin{align*}
bd(t) &= \mu_b + a_b \cdot (bd(t - 1) - \mu_b) + \varepsilon_b(t) \\
\varepsilon_b(t) &= \sigma_b \cdot z_b(t) \\
\sim z_b(t) &\sim N(0, 1)
\end{align*}
9. Short-term Interest Rates

<table>
<thead>
<tr>
<th>( \text{bd}(t) )</th>
<th>1962-2018</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_b )</td>
<td>0.1568 (0.0596)</td>
</tr>
<tr>
<td>( a_b )</td>
<td>0.5527 (0.1116)</td>
</tr>
<tr>
<td>( \sigma_b )</td>
<td>0.1996 (0.0189)</td>
</tr>
<tr>
<td>( r_z(1) )</td>
<td>-0.095</td>
</tr>
<tr>
<td>( r_{z^2}(1) )</td>
<td>0.098</td>
</tr>
<tr>
<td>skewness ( \sqrt{\beta_1} )</td>
<td>-0.2012</td>
</tr>
<tr>
<td>kurtosis ( \beta_2 )</td>
<td>3.1408</td>
</tr>
<tr>
<td>Jarque-Bera ( \chi^2 )</td>
<td>0.4318</td>
</tr>
<tr>
<td>( p(\chi^2) )</td>
<td>0.8058</td>
</tr>
</tbody>
</table>

- Clearly connected with LT interest rates
- Log-ratio is modelled as AR(1)
- All parameters are significant
- Residuals are independent and normally distributed
- No ARCH effect
10. Index-Linked Bonds

\[
\delta_r(t) = \mu_r + a_r.(\delta_r(t-1) - \mu_r) + c_r\delta_c(t) + b_r\delta_b(t) + \epsilon_r(t)
\]

\[
\epsilon_r(t) = \sigma_r.z_r(t)
\]

\[
z_r(t) \overset{iid}{\sim} N(0, 1)
\]

\(\delta_r(t)\): index-linked bond rates
\(\delta_c(t)\): long-term interest rates
\(\delta_b(t)\): short-term interest rates
# 10. Index-Linked Bonds

## 2000-2018

<table>
<thead>
<tr>
<th></th>
<th>$\delta_r(t)$</th>
<th>AR(1)</th>
<th>$\delta_c(t)$ and $\delta_b(t)$</th>
<th>$\delta_c(t)$</th>
<th>$\delta_b(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_r$</td>
<td>0.0222 (0.0033)</td>
<td>0.0438 (0.0316)</td>
<td>-0.0118 (0.0266)</td>
<td>0.0038 (0.0071)</td>
<td></td>
</tr>
<tr>
<td>$a_r$</td>
<td>0.7194 (0.0618)</td>
<td>0.5942 (0.0957)</td>
<td>0.6877 (0.0745)</td>
<td>0.6174 (0.0698)</td>
<td>0.6165 (0.0703)</td>
</tr>
<tr>
<td>$c_r$</td>
<td>-0.2721 (0.1582)</td>
<td>0.1163 (0.0889)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b_r$</td>
<td></td>
<td>0.2142 (0.0727)</td>
<td>0.0973 (0.0419)</td>
<td>0.1144 (0.0272)</td>
<td></td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>0.0033 (0.0004)</td>
<td>0.0038 (0.0007)</td>
<td>0.0034 (0.0004)</td>
<td>0.0029 (0.0003)</td>
<td>0.0030 (0.0003)</td>
</tr>
<tr>
<td>Log Likelihood</td>
<td>77.10</td>
<td>74.61</td>
<td>76.95</td>
<td>79.46</td>
<td>79.32</td>
</tr>
<tr>
<td>$r_z(1)$</td>
<td>0.038</td>
<td></td>
<td></td>
<td>-0.068</td>
<td>-0.060</td>
</tr>
<tr>
<td>$r_{zz}(1)$</td>
<td>-0.072</td>
<td></td>
<td></td>
<td>-0.230</td>
<td>-0.151</td>
</tr>
<tr>
<td>skewness $\sqrt{\beta_1}$</td>
<td>-0.2749</td>
<td></td>
<td></td>
<td>-0.3155</td>
<td>-0.3418</td>
</tr>
<tr>
<td>kurtosis $\beta_2$</td>
<td>2.2125</td>
<td></td>
<td></td>
<td>2.2151</td>
<td>2.3209</td>
</tr>
<tr>
<td>Jarque-Bera $\chi^2$</td>
<td>0.7303</td>
<td></td>
<td></td>
<td>0.8030</td>
<td>0.7349</td>
</tr>
<tr>
<td>$p(\chi^2)$</td>
<td>0.6941</td>
<td></td>
<td></td>
<td>0.6693</td>
<td>0.6925</td>
</tr>
</tbody>
</table>
11. Future Work

- **Real yield curve model**
  - daily index-linked bond rates for 2000-2018
  - maturities – vary from 1 to 20 years, even longer
    - a descriptive yield curve model to fill the missing values in the data
    - applying principal component analysis to obtain uncorrelated variables – level, slope, curvature
    - exploring bi-directional relation
12. Conclusions

- Developed a stochastic investment model for SA actuarial applications based on integrated economic series

- Updated data set and extends the model proposed by Thomson which had a number of practical limitations

- Model has long-term applications for pension funds and life insurance companies

- Several models compared based on statistical tests for independence and normality

- Economic theory incorporated

- Models fit well and the residuals are normally distributed (long-term interest rates!)

- Parameter stability should be analysed in detail

- Yield-curve models can be combined with the other economic series


