

A closed-form approximation for deriving expected losses in excess loss life reinsurance

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ABSTRACT


Primary life insurers need to calculate life reinsurance recoverables for excess-of-loss life reinsurance treaties for solvency purposes as in Solvency II. However, assuming deterministic mortality, the recoverables of excess-of-loss treaties could be zero because the surviving lives are too few to trigger the excess-of-loss barrier. Resorting to simulation may be cumbersome as it may call for blending into a deterministic mortality model such as those of commercial vendors. In this paper we describe an alternative method to avoid simulation that is fast and accurate and can easily be blended into existing commercial software. The results can be used in many instances such as supervisory reporting, reinsurance pricing and risk management.

KEYWORDS

Life reinsurance, excess loss treaties, Lyapunov central limit theorem

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1. INTRODUCTION

1.1 This paper presents a straightforward approach to approximating expected losses in excess loss life reinsurance using a closed-form solution based on Lyapunov's central limit theorem. This approach may be used in the calculation of the best estimate of excess loss life reinsurance liabilities and reinsurance recoverables.

1.2 The paper originates from a practical problem that the author faced when he was working as a life actuarial analyst at a European primary insurer. The insurer had a number of excess-of-loss life contracts and was transitioning to a Solvency II basis for analysing and reporting risks. The need in that case was the calculation of the expected reinsurance recoverables for the block of life reinsurance treaties. Proportional reinsurance recoverables can easily be calculated by multiplying the exposure by the proportion ceded to the reinsurer.

1.3 However, excess-of-loss life treaties present a challenge. Most commercial software assumes that mortality is deterministic. The life runoff under this assumption exhibits no volatility around the mean and the mortality probabilities, q_x , behave as percentages. If the runoff has many lives outstanding the expected loss may be higher than the excess-of-loss barrier. For example, assume 100 lives, with $q_x = 1\%$ and \$1 of sum assured and an excess-of-loss barrier of \$0.5. The expected loss for the 100 lives is \$1, higher than \$0.5, and the life reinsurance has some recoverable value. But as time goes by the pool of survivors reduces. With the number of survivors at 40 the expected loss is \$0.4, lower than \$0.5, and the excess-of-loss reinsurance is worthless. But this is the deterministic view when the probability equals a proportion and does not take into account volatility around the mean. This reinsurance treaty even for 40 survivors is not worthless since there is a small but non-zero probability that five will die and the reinsurance treaty will be triggered.

1.4 One way to solve this problem is through simulation. However, this approach requires computational resources which can be significant for large portfolios. Moreover, in systems where mortality is deterministic it can be cumbersome to include stochastic simulations which imply programming effort and more runs. Especially for companies without large computational resources this can be a problem. The challenge is to find a calculation method beyond the deterministic model, which, in addition, can be easily implemented in commercial software or a spreadsheet.

1.5 One method would be to find a closed-form solution for the calculation in order to avoid simulation. To do that, we use Lyapunov's central limit theorem as an approximation to the distribution of the losses stemming from the excess-of-loss treaty. The Lyapunov central limit theorem states that under some conditions the standardised sum of independent but not identically distributed random variables with finite mean and variance converge to a standard normal distribution. Therefore, we can find the limiting distribution of the total losses of the reinsurance contract and calculate its expected value. Using this approach

requires breaking the underlying insurance contracts into a sequence of annual loss variables as in the individual risk model (see Kaas et al., 2008). These loss variables are of Bernoulli type, and their standardised sum can be approximated by the normal distribution.

1.6 One of the contributions of this paper is to prove formally the application of the Lyapunov central limit theorem for life portfolios, which simplifies a vast array of calculations by aggregating individual losses into a portfolio loss. This technique may be applied to other more complicated reinsurance treaties or as a first step for other more complicated models. For example, if mortality can be explained by a simple distribution, then multidimensional simulations that include mortality can drop one dimension and save computational resources.

1.7 The practical use of this formula is that it is analytically tractable and simple, and can be implemented easily even in a spreadsheet with limited computational resources. The actuary or risk manager can avoid stochastic simulation, especially in cases of primary insurers where life reinsurance recoverables must be reported for supervisory purposes.

1.8 Similar research has not been found in the available literature. Smart (1970) makes the argument for a portfolio approach to life reinsurance but does not pin down an explicit solution. Broader papers cover topics about reinsurance in general and catastrophic reinsurance in particular. Ekhedden & Hoessjer (2014) demonstrate the calculation of catastrophic risk with a Peaks over Threshold model for catastrophic life (re)insurance but it is rather difficult to implement since it hinges on Extreme Value Theory and requires extensive data sets for its calibration. Conversely, the model in this paper is simple, easy to integrate within current Enterprise Risk Management systems and sufficient for supervisory reporting. There is also significant literature covering reinsurance pricing. Refer to Walhin et al. (2001), for example, for a general description of methods and recommended papers on the subject. For an exposition of Extreme Value Theory refer to McNeil et al. (2005).

1.9 In the two next sections, we set up the model and validate the specification via simulation. Section 4 presents a numerical example. Section 5 concludes.

2. Model Setup

2.1 Assume a random population of $i \in \mathbb{N}$ life policyholders with individual coverage amounts C_i and independent death probabilities q_{x_i} at age x_i at a given point in time.¹ Each year, every insurance policy results in a binomial pay-off. Either insured life *dies* with a specific probability q_{x_i} , and the primary insurer pays C_i , or the insured life survives with

1 Since the model is specified in discrete time, the portfolio of insurance contracts could be seen as one cohort with the same starting time in order to eliminate overlapping time periods and keep the model simple without loss of generality. This is important in the multiperiod specification described later in the paper, which includes conditional survival rates.

probability $1 - q_{x_i}$, and the primary insurer pays nothing. The loss of each individual at each period of time is described by a random variable of Bernoulli type:

$$L_i = \begin{cases} C_i, & q_{x_i} \\ 0, & 1 - q_{x_i} \end{cases} \quad (1)$$

2.2 For an excess loss reinsurance contract with loss threshold K , the expected loss for the reinsurer for the whole portfolio for an annual reinsurance contract is defined as:

$$\mathbb{E} \left[\left(\sum_{i=1}^N L_i - K \right) \mathbf{I} \left(\sum_{i=1}^N L_i \geq K \right) \right]$$

with indicator function:

$$\mathbf{I} \left(\sum_{i=1}^N L_i \geq K \right) = \begin{cases} 1, & \sum_{i=1}^N L_i \geq K \\ 0, & \sum_{i=1}^N L_i < K. \end{cases}$$

Further calculations imply:

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{i=1}^N L_i - K \right) \mathbf{I} \left(\sum_{i=1}^N L_i \geq K \right) \right] &= \mathbb{E} \left[\left(\sum_{i=1}^N L_i \right) \mathbf{I} \left(\sum_{i=1}^N L_i \geq K \right) - K \mathbf{I} \left(\sum_{i=1}^N L_i \geq K \right) \right] \\ &= \mathbb{E} \left[\left(\sum_{i=1}^N L_i \right) \mathbf{I} \left(\sum_{i=1}^N L_i \geq K \right) \right] - K \mathbb{E} \left[\mathbf{I} \left(\sum_{i=1}^N L_i \geq K \right) \right] \end{aligned} \quad (2)$$

2.3 Equation (2) stems from the fact that the expectation is a linear operator. The pay-off to the cedant along the continuum of cumulative losses from the underlying portfolio is the same as that of a call option on the loss threshold as strike price (Figure 1).

2.4 If the probabilities of death and the coverage amounts of insured lives in the insurer's portfolio were identically and independently distributed, then the specification of expected total losses using the binomial distribution in equation (2) above would be correct. However, if ages and coverage amounts vary within the portfolio, then the binomial approach no longer holds. Since the Bernoulli random variable can be shown to satisfy the Lyapunov's central limit theorem (Appendix 1), the standardised total losses (i.e. the sum of the Bernoulli random variables) of the underlying portfolio converge, without any reinsurance, to a standard normal distribution. The mean μ and the variance σ^2 of the total losses without reinsurance are given by the following formulas:

$$\mu = \mathbb{E} \left[\sum_{i=1}^N L_i \right] = \sum_{i=1}^N \mathbb{E} [L_i] = \sum_{i=1}^N C_i q_{x_i} \quad (3)$$

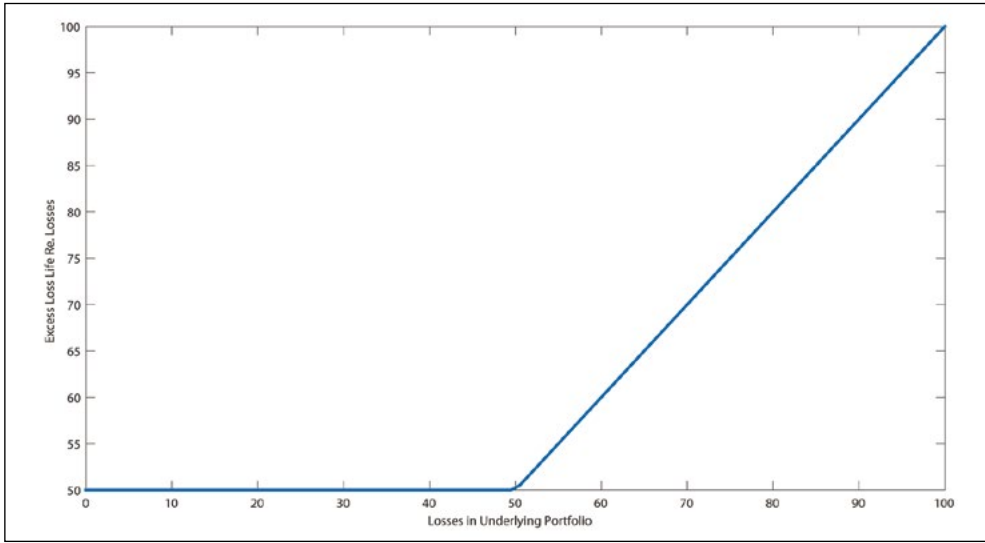


FIGURE 1. Pay-off of excess loss reinsurance contract

$$\sigma^2 = Var \left[\sum_{i=1}^N L_i \right] = \sum_{i=1}^N Var [L_i] = \sum_{i=1}^N \mathbb{E} [L_i^2] - (C_i q_{x_i})^2 = \sum_{i=1}^N C_i^2 q_{x_i} (1 - q_{x_i}) \quad (4)$$

2.5 Re-arranging equation (2) and standardising the total loss we can write equation (2) as:

$$\begin{aligned} \sigma \mathbb{E} \left[\left(\frac{1}{\sigma} \sum_{i=1}^N L_i - \frac{\mu}{\sigma} \right) \mathbf{I} \left(\frac{1}{\sigma} \sum_{i=1}^N L_i - \frac{\mu}{\sigma} \geq \frac{K - \mu}{\sigma} \right) \right] - (K - \mu) \mathbb{E} \left[\mathbf{I} \left(\frac{1}{\sigma} \sum_{i=1}^N L_i - \frac{\mu}{\sigma} \geq \frac{K - \mu}{\sigma} \right) \right] &\rightarrow \\ \sigma \mathbb{E} \left[Z \mathbf{I} \left(Z \geq \frac{K - \mu}{\sigma} \right) \right] - (K - \mu) \mathbb{E} \left[\mathbf{I} \left(Z \geq \frac{K - \mu}{\sigma} \right) \right] & \end{aligned}$$

2.6 Z is a standard normal random variable and represents the limit of the standardised

loss $\frac{1}{\sigma} \sum_{i=1}^N L_i - \frac{\mu}{\sigma}$ if N is sufficiently large. The first quantity $\mathbb{E} \left[Z \mathbf{I} \left(Z \geq \frac{K - \mu}{\sigma} \right) \right]$ is a truncated expectation and is calculated as:

$$\int_{\frac{K - \mu}{\sigma}}^{\infty} z \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{z^2}{2} \right) dz = \frac{1}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} \left(\frac{K - \mu}{\sigma} \right)^2 \right)$$

2.7 Since $Pr\left(Z \geq \frac{K - \mu}{\sigma}\right) = \mathbb{E}\left[\mathbf{I}\left(Z \geq \frac{K - \mu}{\sigma}\right)\right]$, we can derive the expected loss for the threshold choice K as:

$$\frac{\sigma}{\sqrt{2\pi}} \exp\left(-\frac{(K - \mu)^2}{2\sigma^2}\right) - (K - \mu)Pr\left(Z \geq \frac{K - \mu}{\sigma}\right), Z \sim \Phi(0,1). \quad (5)$$

2.8 Equation (5) above has an insurance interpretation. Under the assumption of the overall loss experience converging to the standard normal distribution, the reinsurer will pay one standard deviation of losses, σ , multiplied by the probability of losses reaching the threshold, which is the density function of the normal distribution, less the payments made by the primary insurer $K - \mu$. The payments of the primary insurer are multiplied by the probability that a loss above the threshold of the reinsurance trigger will be realised. Otherwise, had the reinsurance trigger threshold not been exceeded, the primary insurer would have paid on an expected basis $\mu < K$ and not $K - \mu$.

2.9 The specification above is based on a single-period approximation of expected losses but can be extended to a multiperiod set-up by changing the run-off population for each time period over the forecast horizon. In practical terms, the insured lives must be sorted into demographic brackets and an average coverage amount calculated for each bracket. Alternatively, a finer sorting could be established with both ages and coverage amounts varying. This two-dimensional vector would then be projected in run-off by multiplying the current population at a specific age by the corresponding survival probability to project the composition of the population one period ahead.

3. MODEL VALIDATION

3.1 In this section, we validate the proposed model approach by examining the convergence of the closed-form approximation equation (5) above and the actual distribution of losses. One important question is how many underlying life insurance contracts are required for the approximation to be valid.

3.2 To check convergence, we perform a Monte Carlo estimation of a life insurance portfolio with increasing number of insured lives. With this numerical experiment we are comparing the Monte Carlo estimate of the expected loss with the loss that is estimated in closed-form from equation (5).

3.3 We simulate first the insured population with randomly distributed coverage levels and survival probabilities and next the mortality of the individual lives covered by the desired number of insurance contracts. If we seek 100 contracts, for example, we model 100 lives, each with unique mortality probability and sum assured.

3.4 Second, for each life with known death probability and sum assured, we simulate 5 000 trials regarding its mortality probability in order to simulate the random variable in equation (1). To ensure convergence for the simulation of the loss variable (1) we need 5 000 trials. Having carried out in total 500 000 simulations for 100 lives we calculate the expected loss by averaging the number of events out of the 500 000 simulations that produced a loss. We record the loss calculated from the Monte Carlo estimation and check whether the simulated loss has surpassed the threshold K .

3.5 Even if $K=0$, where there is no reinsurance, the expected loss without reinsurance μ cannot be recovered explicitly from equation (5) for any number of underlying life insurance contracts.² We can judge the quality of approximation, however, as the numbers of underlying life insurance contracts increases when $K=0$.

3.6 Figure 2 shows that convergence occurs quickly, after approximately 150 underlying life insurance contracts the approximation formula converges to the Monte Carlo estimate.

3.7 We repeat the process for $K > 0$ (i.e. with reinsurance) for different numbers of contracts and in Figure 3 we see the comparison of the results from the closed-form expression (5) with those obtained from Monte Carlo estimation. In this case, convergence is slower than without reinsurance, and up to 200 contracts are required to validate the convergence. A crucial issue

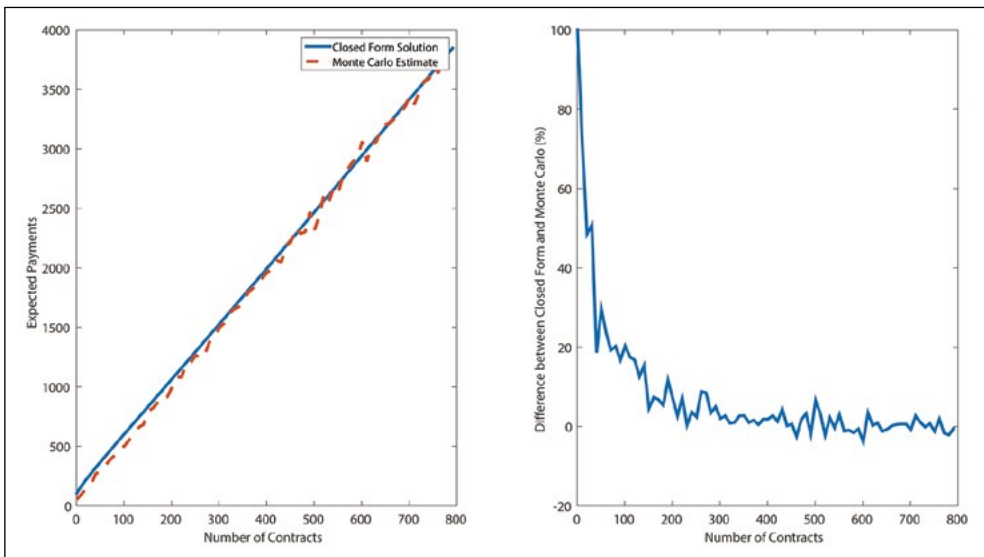


FIGURE 2. Quality of approximation (no reinsurance $K=0$)³

- 2 It is an approximation result which depends on a large number of underlying contracts.
- 3 This chart shows the expected loss derived from the simulation of a life insurance portfolio which is not reinsured and compares it with formula (5) when $K=0$ (i.e. not reinsured portfolio)

is that the number of iterations per life in the Monte Carlo simulation has to be sufficiently high (i.e., more than 5000 trials per contract).⁴

4. NUMERICAL EXAMPLE

4.1 This section provides a numerical example of the application of the formula for the regulatory capital assessment of mortality risk with excess loss reinsurance under Solvency II. Let us assume that a primary insurance company has two life reinsurance treaties, of which excess losses have been ceded for thresholds of $K=3\,000$ and $K=2\,000$, respectively. The company has to calculate its life reinsurance recoverables. The mirror image is the reinsurer who sells these treaties and has to calculate expected losses. In addition, the insurance company has to calculate regulatory solvency capital. Under Solvency II the regulatory solvency capital is the difference in the net asset value (NAV) of the insurer that results from an increase to the death probability q_x of 15 percent, to $1.15q_x$. Since reinsurance recoverables are recorded as assets, the company also needs to increase the mortality rates for the reinsurance treaties in order to correctly capture the hedging effect of reinsurance.

4.2 To show this, we generate two underlying life insurance portfolios for each treaty by randomly drawing for each one 1 000 lives with randomly generated sum assureds and ages.

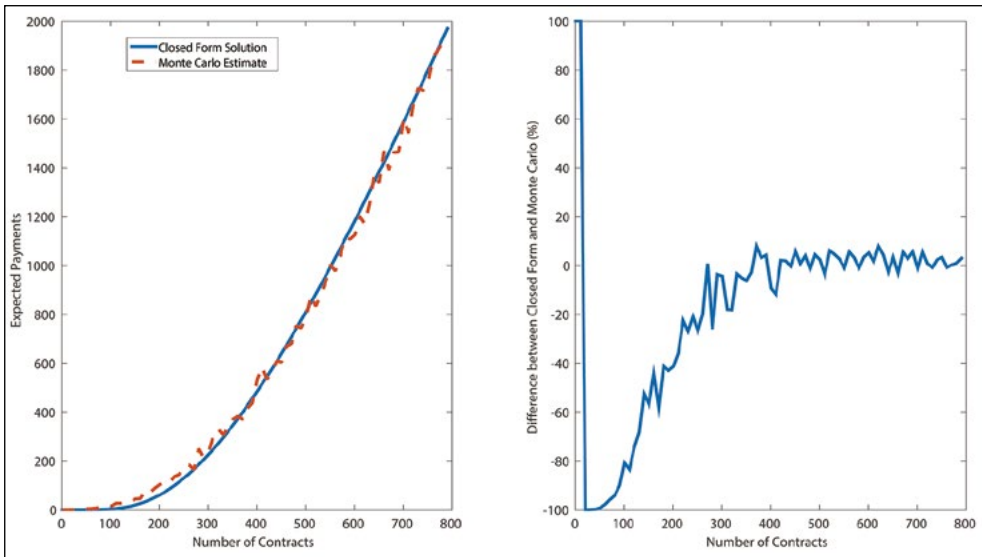


FIGURE 3. Quality of approximation (with reinsurance $K>0$)⁵

4 The closed-form formula needs seconds to calculate the expected losses whereas the Monte Carlo simulation requires close to 15 minutes for a portfolio of 5000 contracts on a desktop PC.
 5 This chart shows the expected loss derived from the simulation of a life insurance portfolio which is reinsured and compares it with formula (5) when $K>0$, (i.e. reinsured portfolio)

For each simulated life $i=1,\dots,1000$ we have a corresponding sum assured C_i and death probability q_{x_i} for each age x_i . Thus, the first two moments (mean μ and variance σ^2) of the total loss distribution, without reinsurance, for each treaty can be calculated using equations (3) and (4). Once these quantities have been calculated, they are plugged into equation (5), which generates expected payments for Treaty #1 and Treaty #2 of 417 and 1712, respectively.

4.3 The results of the calculation are shown in Table 1, which shows the base scenario with the expected reinsurance recoverables and the shocked scenario with reinsurance recoverables under the assumption of a mortality base rate at 1.15 times base scenario.

TABLE 1. SCR calculation of mortality risk

	Basis scenario		Shocked scenario	
	Mean (equation 2)	Std deviation (equation 3)	Mean	Std deviation
Treaty #1	2 885	1 184	3 318	1 269
Treaty #2	3 617	1 445	4 160	1 549
Threshold #1	3 000		3 000	
Threshold #2	2 000		2 000	
Expected payments contract #1	417*		681	
Expected payments contract #2	1 712*		2 217	
Total expected payments	2 130		2 898	
*(equation 4)				

4.4 The difference between the total expected payments in those two scenarios would be a hedge benefit for the insurer since this is a reinsurance asset. For the reinsurer, the difference would be the mortality solvency capital requirement (SCR).

4.5 This calculation does not take into account the volatility of mortality that could result from small samples as noted in a study by the International Actuarial Association (2004), which finds that best estimates are very volatile for small samples. In the overall SCR calculation, the capital charge for mortality risk would then be subject to the diversification effect from combining market risk, non-life risk, credit and health risk.

4.6 Usually multi-year projections are required, especially if the life reinsurance treaty is in force for more than one year. To show how to perform multi-year projections we have the following numerical example. We assume 11 000 insured lives arranged in age brackets with a total sum assured for each age bracket. For each age bracket there is a corresponding survival probability $p_x = 1 - q_x$ for age x . For our calculations we have used the 2008 German mortality table.

TABLE 2. Population of insured lives

Number of contracts	Sum assured	Age	Survival probability
5 000	100	30	99.97%
1 000	50	35	99.96%
2 000	20	40	99.93%
3 000	10	44	99.88%

4.7 At the present the expected payments of the reinsurance treaty are based on the current population. However, one period ahead some lives have died and some have survived. Mortality rates for the survivors are not q_x but q_{x+1} .

TABLE 3. One-period forward projection of mortality risk

Multiperiod projection							
Current year portfolio							
Number of contracts	Sum assured	Age	Survival probability	Mean	Variance	K	Expected payments
5 000	100	30	99.97%	219.68	15 604.10	220	49.68
1 000	50	35	99.96%				
2 000	20	40	99.93%				
3 000	10	44	99.88%				
One-year forward portfolio							
Number of contracts	Sum assured	Age	Survival probability	Mean	Variance	K	Expected payments
$5000 \times 99.97\% = 4998.62$	100	31	99.97%	242.82	17 372.16	220	64.79
$1000 \times 99.96\% = 999.65$	50	36	99.97%				
$2000 \times 99.92\% = 1998.56$	20	41	99.92%				
$3000 \times 99.88\% = 2996.53$	10	45	99.87%				

4.8 The mean total loss without reinsurance μ is calculated by applying equation (3) to each age- and sum-assured bracket.

4.9 The calculated mean and variance of the total loss, along with the threshold $K=220$ are inserted into equation (5). The result is 49.68 expected payments for the life reinsurance treaty. In order to project the population one year forward we multiply the current population with the current survival probabilities.⁶

6 We see that in the first column of Table 3 where the initial population is reducing, the survival probabilities change with the increased age. For example, a life aged 40 has a 99.93% probability of survival for one year. If he or she survives to reach 41, the corresponding probability for the next year is 99.92%.

4.10 For the next projected year, we use the survival probabilities for ages one year forward and we recalculate the mean μ and variance σ^2 of the portfolio loss using equations (3) and (4). The expected payments are now 64.79 one year forward.

4.11 In addition, one can now calculate the two-year best estimate of expected payments at a discount rate of 1% as: $49.68 + 64.79/1.01 = 113.82$.

5. CONCLUSION

5.1 This paper presents a straightforward approach to deriving expected losses in excess loss life reinsurance using a closed-form solution based on Lyapunov's central limit theorem. The numerical example demonstrates how easily the formula can be applied, significantly reducing the computational effort to derive best estimates of excess loss life reinsurance losses. Moreover, this simple formula can be included easily in any type of commercial actuarial software or spreadsheets without any tweaks or stochastic modelling of non-stochastic variables such as deterministic mortality.

5.2 An important assumption behind this formula is that lives are independent. While this may be true in the majority of cases; it does not capture any dependencies in cases where there could be life-tail risk. This formula is nevertheless useful for plain mortality solvency calculations involving excess loss treaties since in many supervisory regimes the life catastrophe risk is separately calculated in the total mortality SCR.

5.3 For life portfolios that may be characterised by significant correlation between lives, using the formula without introducing an allowance for this dependency would be inappropriate. Such an allowance, however, would lead to a multidimensional integral of q_x . Moreover, the formula ignores any long-term improvements in longevity; thus it is a static model that gives a current snapshot of the risks.

5.4 Using the methodology described in this paper, primary insurers can assess their reinsurance recoverables and calculate the gross best estimate of their liability with ease. Reinsurers, on the other hand, could have a first-hand measure of their liability and use the method as part of a pricing tool as well.

DISCLAIMER

The views of the paper are of the author and they do not represent the Bermuda Monetary Authority.

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APPENDIX 1

Proof of Lyapunov’s central limit theorem for Bernoulli random variables

A.1 Suppose $\{X_1, X_2, \dots, X_N\}$ is a set of independent random variables, each with finite expected value μ_i and variance $\sigma_i^2, i = 1, \dots, N$. Define $s_N^2 = \sum_{i=1}^N \sigma_i^2$.

A.2 If $\delta > 0$, the condition:

$$\lim_{N \rightarrow \infty} \frac{1}{s_N^{2+\delta}} \sum_{i=1}^N \mathbb{E} \left[|X_i - \mu_i|^{2+\delta} \right] = 0 \tag{A1}$$

of Lyapunov’s central limit theorem is satisfied, and $\frac{1}{s_N} \sum_{i=1}^N (X_i - \mu_i) \xrightarrow{d} \Phi(0,1)$

converges to the standard normal distribution as N goes to infinity. We can now examine whether the Lyapunov condition holds for the Bernoulli random variable in equation (A1) for $\delta = 1$. We then have:

$$\begin{aligned} \sum_{i=1}^N \mathbb{E} \left[|L_i - \mu_i|^{2+1} \right] &= \sum_{i=1}^N \left((1-q_{x_i}) |0 - C_i q_{x_i}|^3 + q_i |C_i - C_i q_{x_i}|^3 \right) = \\ \sum_{i=1}^N \left((1-q_{x_i}) (C_i q_{x_i})^3 + (1-q_{x_i})^3 C_i^3 q_{x_i} \right) &= \sum_{i=1}^N C_i^3 q_{x_i} (1-q_{x_i}) (1-2q_{x_i} + 2q_{x_i}^2) = \\ \sum_{i=1}^N \sigma_i^2 C_i (1-2q_{x_i} + 2q_{x_i}^2). \end{aligned} \tag{A2}$$

A.3 We know that the following relationship holds:

$$s_N^3 = \left(\sum_{i=1}^N \sigma_i^2 \right)^{3/2}. \tag{A3}$$

A.4 In order to calculate the limit in equation (A1), we use the sandwich theorem of limits. This theorem states that if the limit of the absolute value of a function is smaller or equal to zero, then the limit of the function itself is zero. Substituting the quantities (A2), (A3) into (A1) we have to calculate:

$$\lim_{N \rightarrow \infty} \left| \left(\sum_{i=1}^N \sigma_i^2 \right)^{-3/2} \sum_{i=1}^N \sigma_i^2 C_i (1-2q_{x_i} + 2q_{x_i}^2) \right| \leq \lim_{N \rightarrow \infty} \left| \left(\sum_{i=1}^N \sigma_i^2 \right)^{-3/2} \left| \sum_{i=1}^N \sigma_i^2 C_i \right| \right|. \tag{A4}$$

A.5 The inequality in (A4) stems from the corresponding inequality:

$$1 - 2q_{x_i} + 2q_{x_i}^2 \leq 1, q_{x_i} \in [0, 1].$$

Assuming that $0 < C_i \leq M < \infty$ for all N , we have:

$$\lim_{N \rightarrow \infty} \left(\sum_{i=1}^N \sigma_i^2 \right)^{-3/2} \left| \sum_{i=1}^N \sigma_i^2 C_i \right| \leq \lim_{N \rightarrow \infty} \left(\sum_{i=1}^N \sigma_i^2 \right)^{-3/2} \left| \sum_{i=1}^N \sigma_i^2 M \right| = \lim_{N \rightarrow \infty} \frac{M}{\sqrt{\sum_{i=1}^N \sigma_i^2}} = 0$$

and

$$\lim_{N \rightarrow \infty} \frac{1}{s_N^3} \sum_{i=1}^N E \left[\left[X_i - \mu_i \right]^3 \right] = 0.$$

according to the sandwich (or squeeze) theorem of limits.