

# MODELLING THE MARKET IN A RISK-AVERSE WORLD: THE CASE OF SOUTH AFRICA

By RJ Thomson

## ABSTRACT

In this paper, descriptive models of real returns on the South African market portfolio are developed and analysed. The ‘market portfolio’ is taken to comprise listed equity and government bonds, aggregated in proportion to their market capitalisation from time to time. The models have the attributes that, conditionally on information at the start of a year:

- the real return on the market portfolio during that year is normally distributed; and
- the market price of risk during that year is reasonably greater than zero.

For the purpose of predictive modelling, the best of the models considered was found to be a linear function of the risk-free rate. For that purpose it was decided to use ex-ante estimates of expected returns. This led to bias in the observed mean returns, which negates the rational expectations hypothesis. In the light of the literature on the subject, this is considered acceptable for these purposes.

## KEYWORDS

Market portfolio, risk aversion, South Africa, bias

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## 1. INTRODUCTION

1.1 In Thomson & Gott (2009) a long-term equilibrium model of a local market is developed. As stated in that paper:

“The variables modelled are the prices of risk-free zero-coupon bonds—both index-linked and conventional—and of equities, as well as the inflation rate. The model is developed

in discrete (nominally annual) time, but allowance is made for processes in continuous time subject to continuous rebalancing. It is based on a model of the market portfolio comprising all the above-mentioned asset categories. The risk-free asset is taken to be the one-year index-linked bond. It is assumed that, conditionally upon information at the beginning of a year, market participants have homogeneous expectations with regard to the forthcoming year and make their decisions in mean–variance space.”

1.2 As stated in that paper, that model may be used for the estimation of fair-value prices of liabilities and the determination of benchmarks for the mandating of investment management and the measurement of investment performance. Being an equilibrium model, its use as a pricing model is not confined to contingent claims, but may be extended to the pricing of liabilities in an incomplete market as in Thomson (2005; unpublished b).

1.3 In this paper, alternative models of returns on the South African market portfolio are developed and analysed. The purpose is to find a suitable model of the market portfolio on which to base the equilibrium model in Thomson & Gott (*op. cit.*) or a similar model of the constituents of the market portfolio.

1.4 For the purposes of the paper, ‘returns’ are defined as real annual forces of return. As in Thomson & Gott (unpublished) real returns are used because, in the final analysis, equilibrium must relate to goods and services, not to currencies. As in that paper, the market portfolio is taken to comprise listed equity and government bonds (both conventional and index-linked), aggregated in proportion to their market capitalisation from time to time. The models have the attributes that, conditionally on information at the start of a year:

- the return on the market portfolio during that year is normally distributed; and
- the market price of risk during that year is reasonably greater than zero.

For the purposes of the latter attribute, the market price of risk in year  $t$  is taken to be:

$$R_t = \frac{\mu_{M_t} - \delta_{I_t}}{\sigma_{M_t}} \text{ for } t=1, \dots, N; \tag{1}$$

where:

$$\mu_{M_t} = E \{ \delta_{M_t} | F_{t-1} \};$$

$$\sigma_{M_t}^2 = E \{ (\delta_{M_t} - \mu_{M_t})^2 | F_{t-1} \};$$

- $\delta_{I_t}$  is the real return on a one-year risk-free zero-coupon index-linked bond during the year  $[t-1, t]$ ;
- $\delta_{M_t}$  is the real return on the market portfolio during that year;
- $N$  is the number of years included; and
- $F_t$  is the information at time  $t$ , including  $\delta_{I,t+1}$ .

1.5 In this paper reference is made to ‘descriptive’ and ‘predictive’ models. The distinction between these models follows Thomson (2006a). Briefly:

- “A descriptive model is one that describes historical relationships between the variables modelled.
- “A predictive model is one that predicts future relationships between the variables modelled.”

The models of the market portfolio are initially parameterised as descriptive models. However, the purpose of the development of those models is to inform the definition of a predictive stochastic model for use with the equilibrium model developed in Thomson & Gott (*op. cit.*). As explained in Thomson (2006a), the parameters of a predictive model may be “derived in part from [a] descriptive model, but [may be] adjusted on the basis of other information, theory and the actuary’s subjective judgement.” In the development of the descriptive models, it is therefore borne in mind that the purpose of estimation is to derive ex-post estimates of ex-ante parameters. The rational expectations hypothesis is applied so far as it is possible to do so. However, where that hypothesis conflicts with this purpose, constraints on the estimates are accepted. In the development of the predictive model, adjustments are made to achieve ex-ante estimates.

1.6 By the same token, the role of  $\delta_{It}$  in the models is not primarily to explain the variability of  $\delta_{Mt}$ ; other variables might do so better. It is primarily to satisfy the required attributes. No model of  $\delta_{It}$  is developed in this paper; since (as stated in ¶¶1.1 and 1.4) that variable is the real return on a one-year risk-free zero-coupon index-linked bond, it is modelled as part of the equilibrium model, rather than as part of the market-portfolio model.

1.7 The predictive model envisaged is not intended to constitute ‘the real-world model’ in any unique or logical-positivist sense of that concept. It is merely intended to be a reasonable model for the purposes of ex-ante decision-making. For this reason, as in Thomson & Gott (2009), no hypothesis testing is undertaken, and no out-of-sample tests are made.

1.8 The requirement that the conditional distribution of the return on the market portfolio during some future year (conditional, that is, on information at the start of that year) is normal does not mean that the unconditional distribution of that return will be normal. If, for example, the return in some future year is a function of another variable whose value will be known at the start of that year, but which is not normally distributed in terms of prior information, then the unconditional distribution of the return will not generally be normal. Indeed, it is largely the purpose of this paper to explore the use of models in which the unconditional distribution of the return on the market portfolio is not necessarily normal.

1.9 It also follows that the market price of risk should be reasonably greater than zero not only in the descriptive model, but also for any reasonable value of  $\delta_{It}$  that may occur

in a predictive model. The interpretation of ‘reasonably greater than zero’ is amplified in section 3 below.

1.10 It is envisaged that the predictive model will be parameterised so that the user will not be able to outperform the market on a risk-adjusted basis. This means that the model can be used, for example, to determine market-consistent prices of market-related instruments, and to determine market-consistent liability-based mandates for investment management.

1.11 In section 2, relevant literature is reviewed. This includes literature on risk aversion, on the modelling of the market portfolio and on bias and rational expectations (in particular the estimation of expected returns). In section 3 the models are described. In section 4 the parameterisation of the models is presented and the results are discussed. On the basis of this discussion two candidate models are selected. In section 5 the use of the selected model for predictive purposes is discussed and the model is selected and adapted to allow for ex-ante estimates of expected returns.

1.12 A similar exercise has been undertaken by the author using United Kingdom data. The results are reported in Thomson (unpublished a).

## 2. LITERATURE REVIEW

### 2.1 RISK AVERSION

2.1.1 As pointed out by Merton (1980: 327):

“... a necessary condition for equilibrium is that the expected return on the market must be greater than the riskless rate ... A sufficient condition for this proposition to obtain is that all investors are strictly risk-averse expected utility maximizers.”

2.1.2 For this proposition to obtain, the market price of risk must be positive. While some models of market equilibrium do not rule out a negative market price of risk (e.g. Conrad & Kaul, 1988: 410; Derrig & Orr, 2004: 46), it must be accepted that the long-term financial institutions advised by actuaries (principally life offices and pension funds), effectively being custodians of trust moneys, should be risk-averse. These clients are participants in the process of equilibrium-formation in the capital market. For actuarial purposes, therefore, the models used by actuaries for advising such clients must assume risk-aversion.

2.1.3 Since the publication of the Wilkie (1986) model, numerous stochastic models of returns on assets have been published. Most of these suffer from the drawback that, conditionally on information at the start of a period, they may produce negative market prices of risk during that period. Among the models that may exhibit this phenomenon (not all of which are published in detail), particularly in the case of equities, are:

- the Wilkie (1986, 1995a) model for the United Kingdom (also calibrated for other countries);
- the Carter (1991) model for Australia;

- the Dyson & Exley (1995) model for the U.K.;
- the Thomson (1996) model for South Africa;
- the Harris (unpublished) model for Australia;
- the CAP:Link scenario generation system (Mulvey & Thorlacius, 1998);
- the Boender, Van Aalst & Heemskerk (1998) model for the Netherlands;
- the Whitten & Thomas (1999) model for the U.K.;
- the TY model for the U.K. (Yakoubov, Teeger & Duval, 1999); and
- the Hilli et al (2007) model for Finland.

2.1.4 The Hibbert, Mowbray & Turnbull (unpublished) model for the U.K. avoids this problem.

## 2.2 THE MARKET PORTFOLIO

2.2.1 None of the models listed in the preceding section includes a model of the return on the market portfolio. While they do produce models of major constituents of the market portfolio, their aggregation into a model of the market portfolio would require a model of the composition of that portfolio. The advantage in the explicit modelling of a market-portfolio proxy is that it permits the equilibrium modelling of the various asset categories.

2.2.2 As stated above, in this paper the market portfolio is taken to comprise listed equity and government bonds, aggregated in proportion to their market capitalisation from time to time. As Roll (1977) points out, a model of the market portfolio should include not only equities and bonds, but also all other capital assets, including non-traded assets such as human capital. While this would indeed be required for a true descriptive model, the requirements of a predictive model for decision-making purposes are less exacting, since such a model accommodates the use of subjective assumptions (Thomson, 2006a). Again it must be appreciated that the institutional clients of actuaries invest in a market of traded assets and participate in the process of equilibrium formation within that market. Eun (1994) analyses the capital asset-pricing model (CAPM) into the observable and latent portfolios comprising the market portfolio. He finds that, if the correlation between the two is positive, then, for the observable constituent of the market portfolio, the securities market line has a higher intercept than the risk-free rate. The excess is proportional to the risk premium on the latent constituent. If, however, equilibrium occurs between participants excluded from the latent portfolio, then, for those participants, the intercept must revert to the risk-free rate. This can be accounted for only if the homogeneity of expectations differs as between those participants and others.

2.2.3 The Thomson & Gott (2009) model for South Africa avoids the problem of negative market prices of risk and it includes a model of the market portfolio. However, the specification of the model of the market portfolio in that article was tentative. The exploration of alternative market models was left to further research, which is the subject of this paper.

## 2.3 BIAS AND RATIONAL EXPECTATIONS

2.3.1 The approach adopted in this paper admits the possibility of bias in conditional expected returns on the market portfolio; that is, that ex-ante expectations as represented by the models they use and the values of the parameters in those models may differ from true models and values.

2.3.2 Merton (1980: 125–6) observed that, while substantial effort had been expended on the estimation of the volatilities of returns, little work had been done on expected returns. He suggested that this was due to the relative difficulty of estimating expected returns. However, as Derrig & Orr (*op. cit.*: 46), Campbell (2000: 1522) and Grant & Quiggin (2006) point out, since Mehra & Prescott's (1985) exploration of the 'equity risk-premium puzzle', there have been numerous articles reviewing the expected returns on equity. Conrad & Kaul (*op. cit.*) postulate an autoregressive process for conditional expected return, but their model does not exclude negative market prices of risk. Fama & French (1989) find that expected returns follow a business-cycle pattern and contain a risk premium that is related to longer-term aspects of business conditions. Derrig & Orr (*op. cit.*) document numerous different approaches to the estimation of the equity risk premium, with widely differing results. Wilkie (1995b) contributes yet another. Thomson (2006b) suggests that reference to the equity risk premium 'puzzle' suggests a paradigmatic metanarrative that needs to be deconstructed.

2.3.3 An often unstated assumption underlying the calibration of stochastic models of returns on assets is that the rational expectations hypothesis (REH) (Muth, 1960) holds. While some authors (e.g. Thomson, 1996: 798–9) have cautioned prospective users that their descriptive models may not be appropriate for predictive purposes, the calibration of those models to ex-post observations suggests that, in the absence of information to the contrary, those observations are unbiased estimates of the corresponding ex-ante values.

2.3.4 Numerous studies (Cuthbertson, 1996: 116–201) show that, on certain assumptions, for certain markets at certain times, the REH may be rejected. While many of these relate to short-term effects or to individual shares relative to the market, some of them (e.g. Shiller, 1981; LeRoy & Porter, 1981) are of importance in the long-term modelling of the market. Even in those cases, it has been shown (e.g. Marsh & Merton, 1986) that, with different assumptions, different conclusions may be drawn and Fama (1991: 1586) argues that they do not necessarily reject the REH. Nevertheless, the REH remains questionable. As Cuthbertson (*op. cit.*: 97) points out, tests of the REH that rely on an assumed model such as the capital-asset pricing model (CAPM) involve joint assumptions; rejection does not necessarily imply rejection of the REH. Conversely, however, they would not necessarily imply rejection of the CAPM.

2.3.5 As Roll & Ross (1994) pointed out:

“... a decade of empirical studies [had] reported little evidence of a significant cross-sectional relation between average returns and betas.”

A possible explanation, they suggested, is that market-portfolio proxies are mean-variance inefficient. Another possible explanation is that the REH does not apply. They

refer to the phenomenon as a ‘puzzle’; like the equity risk-premium puzzle, this begs the question whether the paradigm presupposed by the REH is true.

### 3. MODELS OF THE MARKET PORTFOLIO

#### 3.1 INTRODUCTION

3.1.1 In Thomson & Gott (2009; unpublished) a simple model of the return on the market portfolio was adopted, without consideration of more complex but possibly better models. In this section four models are considered: the basic model used in that article, a Markov regime-switching model, an exponential autoregressive (AR) model and an autoregressive conditional heteroscedasticity (ARCH) model. These models are defined below. In choosing the models, the primary consideration was parsimony. The basic model is the simplest model that could be used. Each of the others is a generalisation of that model. Further explanation of the choice of each of those models is explained below.

3.1.2 The data used are the returns on the South African market-portfolio proxy and on the one-year risk-free zero-coupon bond for the period from 1987 to 2008. As noted above the market-portfolio proxy comprises South African listed equity and government bonds (both conventional and index-linked). These were the same data as used in Thomson & Gott (unpublished), except that data for the two years 1987 and 1988 were not used in that paper, nor were those for 2006 to 2008. The reason for the exclusion of 1987 and 1988 from Thomson & Gott (unpublished) was that, in those years, the formula for the risk-free rate (*viz.* the one-year conventional bond yield less the inflation rate) gave negative values. In the first place, this conflicted with the requirement of arbitrage-freedom in that, in principle, a market participant might short a risk-free real asset and hold risk-free contracts for the delivery of goods and services to earn a risk-free profit. Secondly, this phenomenon does not imply that the *ex-ante* risk-free rate was negative; it is arguably just a result of the formula used for the period prior to the issue of index-linked bonds. In this paper the formula has been subjected to a minimum of zero. While this may introduce some bias in that arbitrage may not be achievable in practice, that bias is considered acceptable in the light of the discussion in section 2.3. The data for 2006 to 2008 have become available since the date of that paper.

3.1.3 As mentioned in ¶2.2.2, the market portfolio should include all assets in which the actuary’s client may invest. A notable absence is fixed property. Since many fixed properties are owned by listed companies, it would be necessary to avoid the double-counting involved. Corporate debt should also be included. Until such time as the necessary data are available, the portfolio used in this paper is an approximation to the best proxy available.

3.1.4 The data set is small, comprising only 22 values of each of the variables. It would have been possible to use quarterly data, but for the purpose of annual decision-making that would be of questionable value (Thomson, 1996). As Merton (*op. cit.*) points out, the precision of the estimate of expected returns depends on the total length of calendar time, rather than on the number of observations per se. For consistency, the commencement date of the data set corresponds (with the modification

explained in ¶3.1.2) to that of Thomson & Gott (unpublished). That in turn was dictated by the availability of data. While it may be argued that the resulting data set is too small to permit the credible selection of a market-portfolio model, it must be recognised that, in a rapidly changing world, it is questionable whether long data sets are relevant to the future. This is particularly true in South Africa, where conditions in earlier years were markedly different from those of the present.

3.1.5 In view of the small data set, particular care needs to be taken to avoid the treatment of spurious or fortuitous relationships as important. Also, in specifying models involving autoregressive effects, long lags should not be considered. If such a lag is of greater significance than a shorter lag, the effect would have to be regarded as fortuitous. In this paper only one-year lags are considered.

3.1.6 Since, as explained above, it is not intended that any predictive model based on the descriptive models developed in this paper is uniquely valid, it is considered better to retain reasonable uncertainty in the model than to achieve high levels of likelihood based on fortuitous relationships.

### 3.2 THE BASIC MODEL

3.2.1 As explained in Thomson & Gott (2009; unpublished),  $\mu_{Mt}$  cannot be modelled as a constant because this would result in negative risk premiums from time to time. Instead, as in those papers, following the capital-market line of the CAPM, we may model  $\delta_{Mt}$  as:

$$\delta_{Mt} = g\delta_{It} + h + \sigma_M \varepsilon_t; \tag{2}$$

where:

$$\begin{aligned} &\varepsilon_t \sim N(0,1); \\ &\text{cov}\{\varepsilon_t, \varepsilon_s\} = 0 \text{ for } s \neq t; \\ &g \geq g^* \geq 1; \text{ and;} \\ &h \geq h^* \geq 0; \end{aligned}$$

so that:

$$\mu_{Mt} = g\delta_{It} + h.$$

This formulation is justified as in Thomson & Gott (unpublished: ¶6.2.7) on the grounds of simplicity and on the grounds that the risk premium

$$\pi_t = \frac{\mu_{M,t} - \delta_{I,t}(0)}{\sigma_{M,t}}$$

is positive, though it may vary according to the level of  $\delta_{I,t}(0)$ . Further justification is given in that paper.

3.2.2 The purpose of introducing the constants  $g^*$  and  $h^*$  is to establish lower bounds for  $g$  and  $h$  respectively. In order to avoid negative market prices of risk, we may take  $g^* = 1$  and  $h^* = 0$ . These are referred to below as the ‘basic constraints’. However, the purpose of setting  $g^*$  and  $h^*$  greater than or equal to 0 is to ensure not merely that the market price of risk is non-negative, but also, as explained in section 1, that it is reasonably greater than 0. For this purpose it is required either that  $g^* = 1, 2$  and

$h^* = 0$  or that  $g^* = 1$  and  $h^* = 0,01$ . (The concept ‘reasonably greater’ is necessarily both arbitrary and subjective. In view of the observations in ¶1.5 that is acceptable.) These are referred to below as the ‘required constraints’.

3.2.3 If  $g > 1$  and  $h = 0$  then the risk premium depends proportionately on the level of  $\delta_{It}$ . If  $g = 1$  and  $h > 0$  then it is independent of the level of  $\delta_{It}$ . If  $g > 1$  and  $h > 0$  then the risk premium depends linearly on the level of  $\delta_{It}$ . There is no *prima facie* reason for preferring any one of these three possibilities; hence the inclusion of all three candidates. It should be noted that, while the above constraints ensure that  $\mu_{Mt} \geq \delta_{It}$ , they do not ensure that  $\delta_{Mt} \geq \delta_{It}$ . The latter constraint would be neither realistic nor desirable.

3.2.4 In Thomson & Gott (unpublished) it was found that  $h$  was not significant at the 95% level. With  $h = 0$  an estimate of  $g = 1,7$  was obtained. The 95% confidence limits of  $g$  were 1 and 3,1, so that this estimate was not reliable. It was nevertheless used in that paper for the purposes of illustration.

### 3.3 THE REGIME-SWITCHING MODEL

3.3.1 Another possible approach would be to use a Markov regime-switching model (Hamilton, 1989), with a similar structure in each regime, i.e.:

$$\delta_{Mt} = g_{S_t} \delta_{It} + h_{S_t} + \sigma_{S_t} \varepsilon_t; \tag{3}$$

where:

$$S_t \in \{0, 1\};$$

$$\Pr\{S_t = 0 | S_{t-1} = 0\} = p_{00};$$

$$\Pr\{S_t = 1 | S_{t-1} = 0\} = p_{01} = 1 - p_{00};$$

$$\Pr\{S_t = 0 | S_{t-1} = 1\} = p_{10};$$

$$\Pr\{S_t = 1 | S_{t-1} = 1\} = p_{11} = 1 - p_{10};$$

and  $\varepsilon_t \sim N(0,1)$  is serially independent, so that, conditionally on information at time  $t - 1$ :

$$\delta_{Mt} \sim N(\mu_{Mt}, \sigma_{Mt}^2);$$

where:

$$\mu_{Mt} = g_{S_t} \delta_{It} + h_{S_t};$$

$$\sigma_{Mt} = \sigma_{S_t};$$

$$g_s \geq g_s^* \geq 0; \text{ and}$$

$$h_s \geq h_s^* \geq 1.$$

3.3.2 The choice of this model is suggested by the successes of regime-switching models in the literature (e.g. Hardy, 2001; Harris, unpublished; Maitland, 2010). Also, it permits departure from the normal distribution, particularly the inclusion

of high kurtosis. As for the basic model, it is required, for each  $s$ , either that  $g_s^* = 1, 2$  and  $h_s^* = 0$  or that  $g_s^* = 1$  and  $h_s^* = 0, 01$ . In the basic model, the estimate of  $\sigma_M$  required no constraint. In the regime-switching model, however, one of the regimes may produce an estimate of  $\sigma_s$  that will yield unreasonably large market prices of risk. This is further considered in the parameterisation of the model in section 4.2 below.

3.3.3 As mentioned in section 1 above, it is required that, conditionally on information at the start of a year, the return on the market portfolio during that year be normally distributed. It may appear at first sight that this contradicts the intention of a regime-switching model, viz. that the regime is not known with certainty at any time. However, this intention is not violated if we assume that market participants must make their decisions at the start of a year before the regime is known and that immediately thereafter the regime is known. This is merely a matter of convenience in the framing of a discrete-time model. In order to accommodate this requirement it is assumed that  $F_{t-1}$  includes  $S_t$ ; i.e. that the regime is known at the start of the year. It is for this reason that the parameters must satisfy the required constraints in each regime; otherwise the distribution of the return would have a mixture density.

### 3.4 THE EXPONENTIAL AR MODEL

3.4.1 Many of the stochastic models developed in the actuarial literature are linear autoregressive moving-average (ARMA) time series (e.g. Wilkie, 1986, 1995a; Carter, *op. cit.*; Thomson, 1996). Now  $\mu_{M_t}$  cannot be modelled as an ARMA time series because this would result in negative risk premiums from time to time. However,  $\delta_{M_t}$  may for example be modelled as:

$$\delta_{M_t} = g\delta_{M_t} + h \exp\{\alpha(\delta_{M,t-1} - g\delta_{M,t-1})\} + \sigma_M \varepsilon_t; \tag{4}$$

where:

- $g \geq g^* \geq 1$ ;
- $h \geq h^* \geq 0$ ; and
- $\varepsilon_t \sim N(0,1)$  is serially independent;

so that, conditionally on information at time  $t - 1$ :

$$\delta_{M_t} \sim N(\mu_{M_t}, \sigma_M^2);$$

where:

$$\mu_{M_t} = g\delta_{M_t} + h \exp\{\alpha(\delta_{M,t-1} - g\delta_{M,t-1})\}.$$

Here the absolute value of  $\alpha$  represents the degree to which the previous year's excess return influences  $h$  and its sign indicates whether (in the case of the negative) expected returns are higher after relatively low returns. By expressing this influence as an exponential effect instead of a linear ARMA effect, we avoid the possibility of a negative market price of risk.

3.4.2 Again it is required either that  $g^* = 1, 2$  and  $h^* = 0$  or that  $g^* = 1$  and  $h^* = 0, 01$ . However, if  $h = 0$ , the model reduces to the basic model, so the first constraint falls away.

3.4.3 This model is referred to in this paper as the 'exponential AR model'.

3.5 THE ARCH MODEL

3.5.1 A fourth possibility is to include ARCH effects (Engle, 1982);  $\delta_{Mt}$  may, for example, be modelled as:

$$\delta_{Mt} = g\delta_{ht} + h + z_t;$$

where:

$$\begin{aligned} z_t &= \sigma_t \varepsilon_t; \\ \sigma_t^2 &= a + bz_{t-1}^2; \text{ and} \\ \varepsilon_t &\sim N(0,1) \text{ is serially independent.} \end{aligned} \tag{5}$$

3.5.2 This model is considered because ARCH effects have been studied by a number of authors in the actuarial literature (e.g. Harris, 1994, Hua, unpublished). Also, they indirectly allow the inclusion of high kurtosis. ARCH effects may be modelled in other ways, but in the interests of simplicity, attention has been confined to the model described above.

3.6 GENERAL REMARKS

3.6.1 It may be noted that the basic model is a particular case of each of the other models. The use of the latter models must therefore be justified in terms of their additional descriptive value. It would be possible to devise other such models, but the models chosen have the advantages not only of parsimony and comparability, but also of reasonable track records in the actuarial literature.

3.6.2 For each model, the likelihood function of the model and the maximum-likelihood estimates of the parameters were determined as set out in the appendix. Where, for the purposes of subsequent discussion, it was necessary to do so, the confidence limits of those estimates were also determined.

3.6.3 For the purposes of comparison of the descriptive value of the respective models, the Akaike information criterion (AIC) was used, viz.:

$$A=2k-2l;$$

where:

$k$  is the number of parameters;

$l=\ln(L)$ ; and

$L$  is the likelihood of the observed values (Akaike, 1974).

A model with a lower AIC is preferable.

3.6.4 For each model considered (and, in the case of the regime-switching model, for each regime), the mean market price of risk was calculated, viz.:

$$R = \frac{\hat{\mu}_M - \bar{\delta}_l}{\hat{\sigma}_M};$$

where:

$\hat{\mu}_M$  and  $\hat{\sigma}_M$  are estimates of the mean and standard deviation of  $\delta_{Mt}$  based on the model;

$$\bar{\delta}_l = \frac{1}{N} \sum_{t=1}^N \delta_{ht};$$

$\delta_{Mt}$  is the real return on the market portfolio during year  $[t-1;t]$ ; and  $\delta_{It}$  is the real return on a one-year risk-free zero-coupon bond during that year.

3.6.5 Also, the implied bias in the mean (that is the excess of the sample mean over the long-term mean implied by the model) was calculated, viz.:

$$B = \bar{\delta}_M - \hat{\mu}_M;$$

where:

$$\bar{\delta}_M = \frac{1}{N} \sum_{t=1}^N \delta_{Mt}.$$

3.6.6 Finally, where possible, a Q-Q plot (Wilk & Gnanadesikan, 1968) was produced. For this purpose the set  $\{z_t = \delta_{M;t} - g\delta_{I;t} + h \mid t = 1, \dots, N\}$  was ordered to give  $\{z^{(r)} \mid r = 1, \dots, N\}$  such that:

$$z^{(r)} \geq z^{(r-1)} \text{ for } r = 2, \dots, N.$$

3.6.7 The Q-Q plot was then defined by the points:

$$\left( \Phi^{-1}\left(y^{(r)}\right), z^{(r)} \right);$$

where:

$$y^{(r)} = \frac{r - \frac{1}{2}}{N}; \text{ and} \tag{6}$$

$\Phi(\bullet)$  is the distribution function of the normal variable with mean 0 and standard deviation  $\sigma_M$ .

In equation (6) the numerator and denominator represent the number of observations less than  $z^{(r)}$  and  $z^{(N)}$  respectively, the second term in the numerator being an adjustment for symmetry. For the purposes of comparison of the Q-Q plots the statistic:

$$Q = \sqrt{\frac{\sum_{r=1}^N \left\{ \Phi^{-1}\left(y^{(r)}\right) - z^{(r)} \right\}^2}{N - 1}}$$

was calculated. A model with a lower  $Q$  is preferable.

#### 4. THE PARAMATERISATION OF THE MODELS

In this section the parameterisation of each of the models is presented in sections 4.1 to 4.4 and the results are briefly discussed. In section 4.5 the results are then summarised and compared between the models considered.

##### 4.1 THE BASIC MODEL

4.1.1 For the basic model the likelihood function and the estimates and confidence limits of the parameters were obtained in closed form following (e.g.) Hocking (1996: 136–45) (see section A.1 in the appendix). Table 1 shows the results of the parameterisation. The unconstrained case did not satisfy the basic constraints, so only the constrained cases are shown. The same constraints are applied to the confidence limits as to the estimates.

4.1.2 It may be noted that, in relation to their respective values, the confidence intervals of the parameters are wide, particularly in the cases of  $g$  and  $h$ , though they

have narrowed slightly in comparison with Thomson & Gott (unpublished) because of the extra data. This leaves considerable scope for discretion in the determination of parameters for the predictive use of the model. The AIC is lowest for  $h = 0$  and, for the purposes of this paper, that case is adopted for the purposes of comparison with the other models considered.

Table 1: Parameterisation of the basic model

Parameter	Details	Constraints	
		$g = 1$	$h = 0$
$g$	estimate		1,57
	confidence limits		1; 3,2
$h$	estimate	0,019	
	confidence limits	0; 0,09	
$\sigma_M$	estimate	0,173	0,171
	confidence limits	0,12; 0,22	0,12; 0,22
$k$		2	2
$l$		7,86	8,12
$A$		-11,71	-12,23
$R$		0,11	0,15
$B$		0	-0,007
$Q$			0,0197

4.1.3 Figure 1 plots the observed values, estimates and 95% confidence limits of  $\delta_{Mt}$  against  $\delta_{It}$  for the basic model. (The estimates and confidence limits of  $\delta_{Mt}$  are conditional on the observed values of  $\delta_{It}$ .) Figure 2 shows the same information in time-series form. It is clear that  $\delta_{It}$  explains very little of the variability in  $\delta_{Mt}$ . However, as explained above, this is not the point; the purpose of including  $\delta_{It}$  is not to explain the variability in  $\delta_{Mt}$ , but to ensure that the market price of risk is positive; i.e. that  $\delta_{Mt} > \delta_{It}$ .

4.1.4 The Q–Q plot of the basic model is shown in Figure 3. In view of the small data set, this plot gives credible results.

## 4.2 THE REGIME-SWITCHING MODEL

4.2.1 For the regime-switching model the likelihood function was found following Hamilton (*op. cit.*) (see section A.2 in the appendix.) The maximum likelihood was found by means of the Nelder–Mead algorithm (Nelder & Mead, 1965). The results are shown in Table 2. Confidence limits of the parameters were not calculated.

4.2.2 It may be noted from Table 2 that  $\sigma_0$  is extremely low. This is reflected in the narrow confidence limits of regime 0 in Figure 4, which shows the observed values, estimates and 95% confidence limits of  $\delta_{Mt}$  against  $\delta_{It}$  for the regime-switching model. Nevertheless, the value of  $g$ , and therefore the expected return on the market portfolio, is relatively high in regime 0.

Figure 1: Basic model:  $\delta_{Mt}$  against  $\delta_{It}$

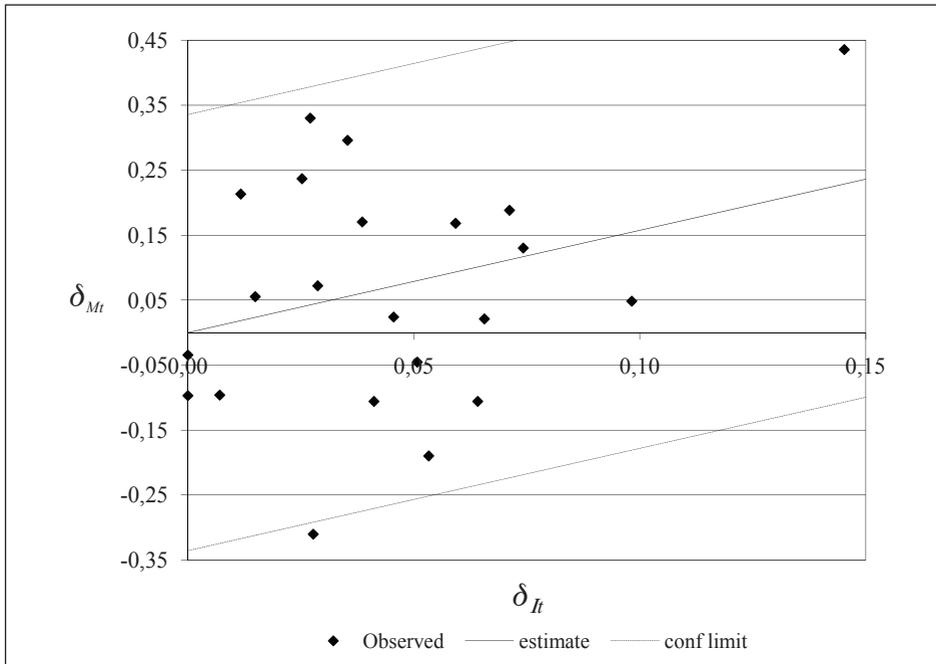
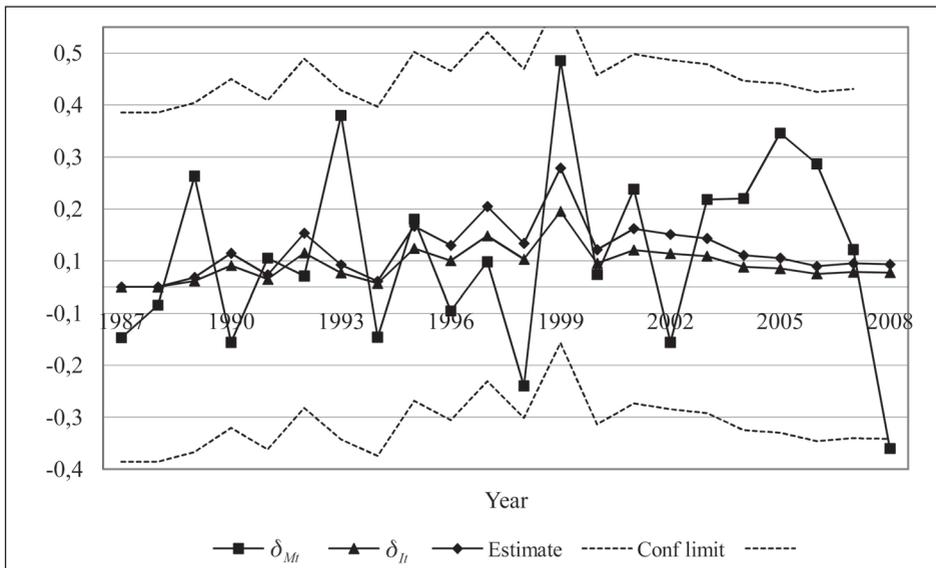


Figure 2: Basic model: time series

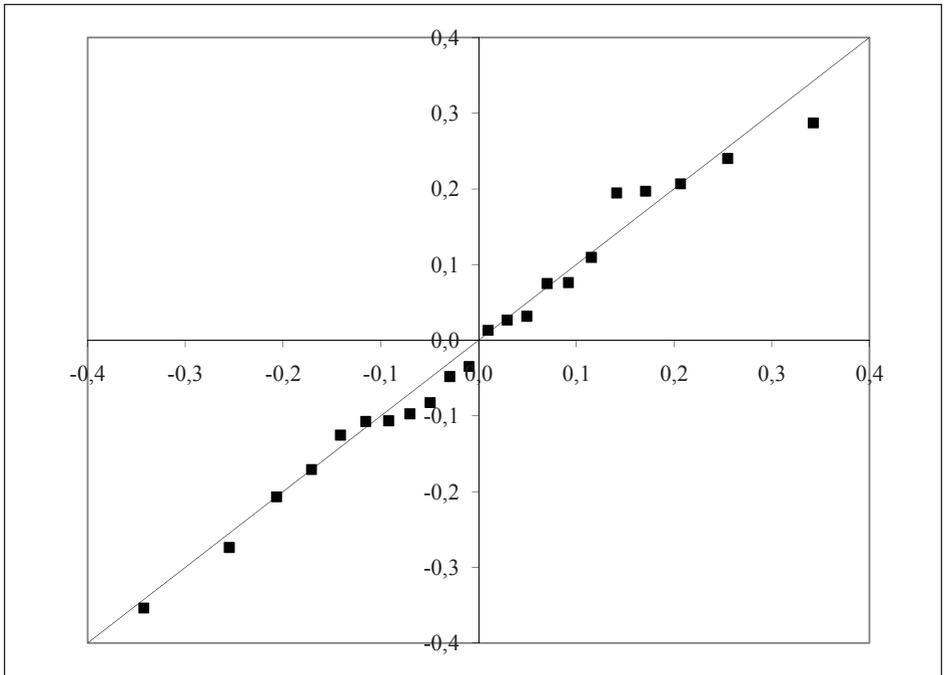


4.2.3 It may be noted from Figure 4 that the parameterisation effectively identifies a fortuitous alignment of about five points. These are associated with regime 0 with high probabilities, while all other points are associated with regime 1 with even higher probabilities. As mentioned in section 3, in view of the small data set, particular care needs to be taken to avoid the treatment of spurious or fortuitous relationships as important. The market price of risk  $R$  is very high; this is largely because of the very low volatility in regime 0. For this reason, the regime-switching model should not be adopted without further analysis; further consideration is given to it in section 4.5 below.

Table 2: Parameterisation of the regime-switching model

Parameter		Parameter	
$p_{00}$	0	$\sigma_1$	0,176
$p_{10}$	0,25	$l$	10,90
$g_0$	2,93	$k$	5
$h_0$	0	$A$	-11,80
$\sigma_0$	0,015	$R$	0,45
$g_1$	1,2	$B$	-0,010
$h_1$	0		

Figure 3: Basic model: Q-Q plot



### 4.3 THE EXPONENTIAL AR MODEL

4.3.1 For the exponential AR model the likelihood function was obtained in closed form as shown in section A.3 in the appendix, but it was not possible to obtain the estimates of the parameters in closed form. As for the regime-switching model, the estimates were obtained by means of the Nelder–Mead algorithm.

4.3.2 Table 3 shows the results of the parameterisation of this model. Because the parameterisation is based on 21 observations instead of 22, the value of the log-likelihood has been multiplied by 22/21. This adjustment is not necessarily accurate, as the contribution of year 1 does not necessarily correspond to the average log-likelihood. As for the basic model, in relation to their respective values, the confidence intervals of the parameters other than  $\sigma_M$  are quite wide. It may be noted that the confidence interval of  $\alpha$  ranges from negative to positive, suggesting that it is not significant. However, because (as explained in §§1.7 and 3.6) the choice of model is based on comparisons of the AIC, the market price of risk, the bias and the Q–Q plots rather than on significance testing, the model is not dismissed at this stage.

4.3.3 Figure 5 plots the observed values, estimates and 95% confidence limits of  $\delta_{Mt}$  in time-series form.

Figure 4: Regime-switching model

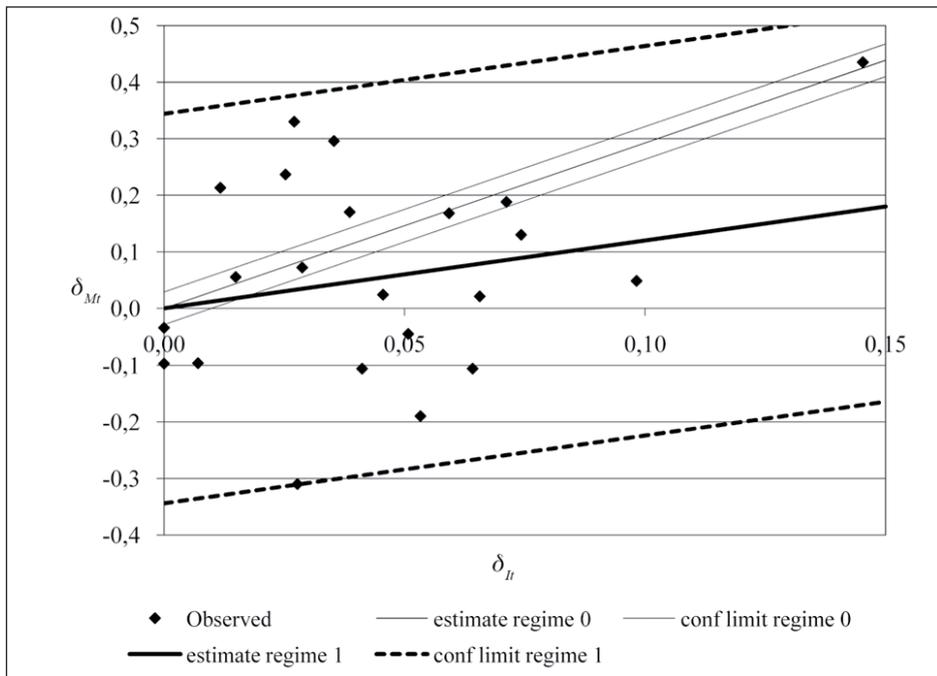
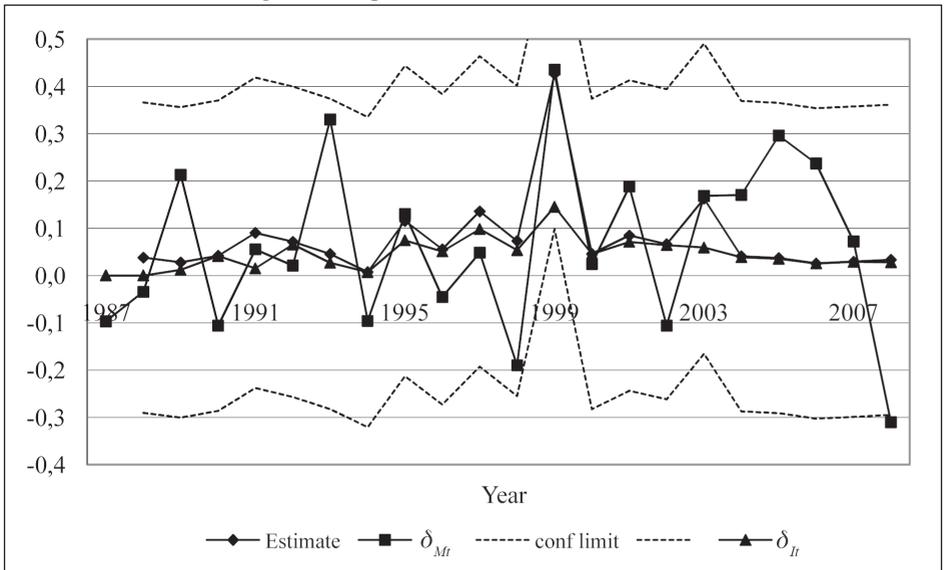


Table 3: Parameterisation of the exponential AR model

Parameter	Details	
$\alpha$	estimate	-13,72
	confidence limits	-15,8; 2,8
$g$	estimate	1
	confidence limits	1; 2,2
$h$	estimate	0,01
	confidence limits	0,01; 0,06
$\sigma_M$	estimate	0,167
	confidence limits	0,12; 0,26
$k$		2
$l$		9,14
$A$		-14,29
$R$		0,21
$B$		0,001
$Q$		0,0238

4.3.4 A Q-Q plot is shown in Figure 6. As for the basic model, the results appear reasonable.

Figure 5: Exponential AR model time series



### 4.4 THE ARCH MODEL

4.4.1 For the ARCH model the maximum-likelihood estimates of  $g$  and  $h$  are the same as those for the basic model. The likelihood function was obtained in closed form as shown in section A.4 of the appendix and the Nelder–Mead algorithm was used to determine the maximum-likelihood estimates of the parameters. The 95% confidence limits of the estimates were determined by simulation.

4.4.2 Table 4 shows the results of the parameterisation of the ARCH model. From that table it may be noted that the estimate of  $b$  is zero, the minimum constraint. From equation (5) it follows that the model reduces to the basic model, with:

$$\sigma_M^2 = (0,171)^2 = 0,029 = a.$$

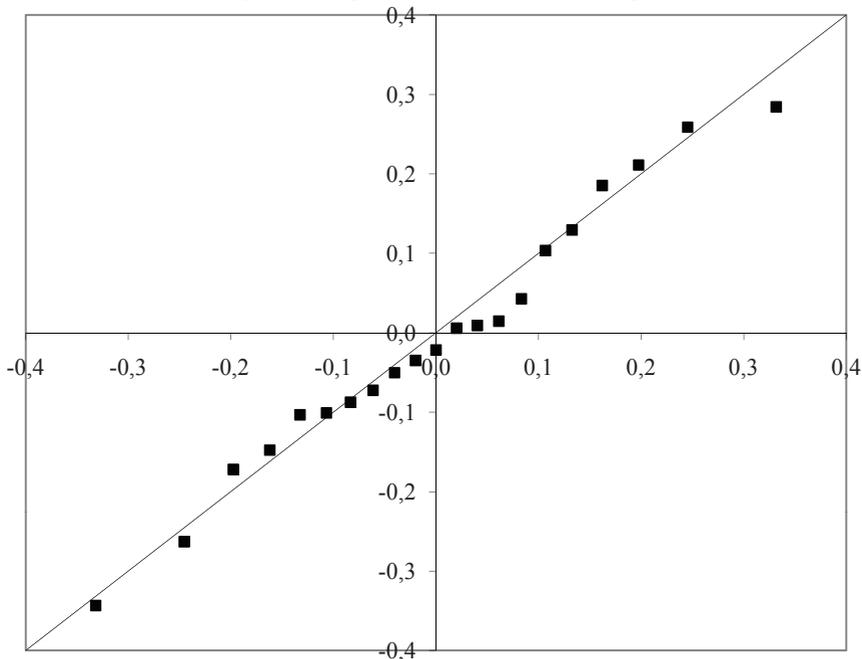
The ARCH model is therefore not further considered.

### 4.5 SUMMARY AND COMPARISON OF THE DESCRIPTIVE MODELS

4.5.1 Table 5 summarises the selection criteria of the descriptive models.

4.5.2 For all three models the bias  $B$  is reasonable, though it is relatively large for the regime-switching model. For the basic model and the exponential AR models the market price of risk  $R$  is reasonable, though somewhat high in the latter case. For the regime-switching model the market price of risk is very high. On the basis of the AIC it would appear that the exponential AR model should be selected. However,

Figure 6: Exponential AR model Q–Q plot



in view of the small sample size, care must be taken to avoid spurious effects. Further analysis of the likelihoods of the basic and exponential models is therefore necessary.

4.5.3 The result  $Q$  of the Q–Q plot calculations shows that the residuals of the basic model give a more credible fit to a normal distribution than the exponential AR model, though the difference is not substantial.

4.5.4 Secondly, it may be noted that, by virtue of the reduction of  $h$  by a factor that may be less than unity (so that  $h < 0,01$ ) the formulation of the exponential AR model relaxes the requirement that the excess return on the market must exceed 0,01. The optimal selection of  $\alpha$  enhances this effect. The fact that the exponential AR model has a lower AIC than the basic model is largely attributable to this effect, so the appearance that this model is preferable to the basic model may be spurious.

4.5.5 Another effect increasing the likelihood of the exponential AR model is the good fit of the latter in 1999. Indeed, it appears that the optimisation of  $\alpha$  is largely to achieve this effect. Excessive sensitivity of a parameter to a single data point must signal some caution to the acceptance of that parameter. Before accepting the exponential AR model it would be advisable to explore this matter further.

Table 4: Parameterisation of the ARCH model

Parameter	Details	
$g$	estimate	1,57
	confidence limits	1; 3,2
$h$		0
$a$	estimate	0,029
	confidence limits	0,025; 0,032
$b$	estimate	0
	confidence limits	0,04
$k$		2
$l$		7,79
$A$		-11,57
$R$		0,15
$B$		0,007

Table 5: Summary of selection criteria

Criterion	Model		
	basic	regime-switching	exponential AR
$A$	-12,23	-11,80	-14,29
$R$	0,15	0,45	0,21
$B$	0,007	0,010	0,001
$Q$	0,0197	not calculated	0,238

4.5.6 Thirdly, as pointed out in ¶4.3.2, the adjustment of the likelihood of the exponential AR model for the purposes of comparison is not accurate.

4.5.7 Furthermore, it may be noted from Table 3 that the 95% confidence interval of  $\alpha$  ranges from  $-15,8$  to  $2,8$ . If  $\alpha = 0$  then the model would reduce to the basic model (with  $g = 1$ ). This value is well within the confidence interval.

4.5.8 While the regime-switching model may be rejected on the grounds of Table 5, the choice between the basic and exponential AR models is not conclusive. Furthermore, as shown in the next section, it is affected by considerations relating to the use of the model for predictive purposes. The choice is therefore further considered in the next section.

## 5. THE USE OF THE MODEL FOR PREDICTIVE PURPOSES

5.1 As explained in section 1, the purpose of the development of the descriptive models in this paper is to derive ex-post estimates of ex-ante parameters. The reason for this is that, for predictive purposes, ex-ante parameters are required. As observed in Thomson (2006), some subjectivity may be required in the selection of ex-ante parameters.

5.2 The question that arises in using the descriptive model to inform the development of a predictive model, is whether and to what extent the biases of the past will persist in the future. A different way of posing the question is: While the market did not follow the ex-ante estimates of the model in the past, is it reasonable to assume that it will do so in the future? The first presupposes that the ex-ante estimates were wrong, while the second presupposes that the sample observed was fortuitously different from the estimates. The question is particularly pertinent in relation to the regime-switching model, which implied a relatively large bias, but it is also relevant to the basic model.

5.3 On the basis of the data, the unbiased ex-post estimate of  $\mu_M$ , i.e.  $\bar{\delta}_M$ , is 0,064.

5.4 For predictive purposes an unbiased ex-ante estimate of  $\mu_M$  is required. This is not necessarily best estimated by ex-post maximum likelihood. As mentioned in section 2.3, the problem of unbiased estimation of ex-ante expected returns has been addressed by numerous authors. Amongst these are the following estimates of the excess return on equities over the risk-free rate (all expressed as annual rates):

- Wilkie (1995b): 3% long-term median on U.K. equities;
- Derrig & Orr (2004): 4% to 5% on U.S. equities;
- Campbell (unpublished): 3,8% on world equities.

If we take the estimate for equity at 3,8%, this converts to an annual force of 3,7%.

5.5 Risk premiums on long-term bond returns are generally lower. If we express the CAPM as:

$$\bar{\mu}_E = \frac{\hat{\sigma}_{EM}}{\hat{\sigma}_{MM}}(\bar{\mu}_M); \text{ and}$$

$$\tilde{\mu}_C(20) = \frac{\hat{\sigma}_{CM}(20)}{\hat{\sigma}_{MM}}(\tilde{\mu}_M);$$

where:

- $\tilde{\mu}_E = 0,037$  is the excess return on equities over the risk-free rate;
- $\tilde{\mu}_C(20)$  is the excess return on a 20-year conventional bond over the risk-free rate;
- $\tilde{\mu}_M$  is the excess return on the market portfolio over the risk-free rate;
- $\sigma_{EM}$  is the covariance between the return on equities and the return on the market;
- $\hat{\sigma}_{CM}(20)$  is the covariance between the return on a 20-year conventional bond and the return on the market; and
- $\hat{\sigma}_{MM}$  is the variance of the return on the market;

then:

$$\tilde{\mu}_C(20) = \frac{\tilde{\mu}_E \hat{\sigma}_{CM}(20)}{\hat{\sigma}_{EM}}. \tag{7}$$

5.6 The 20-year conventional bond is used for simplicity. Index-linked bonds are ignored because of their relatively small market capitalisation. From the South African market-portfolio data described in Thomson & Gott (unpublished), extended to 2008, and using the definitions in that paper, we obtain:

$$\hat{\sigma}_{EM} = 0,030; \text{ and}$$

$$\hat{\sigma}_{CM}(20) = 0,011.$$

Thus, from equation (7):

$$\tilde{\mu}_C(20) = 0,014 .$$

5.7 From the same data, the weighting of bonds to equities in the South African market portfolio as at 31 December 2008 was 0,13:0,87. On the basis of those weights, the excess annual return on the market portfolio may be taken at:

$$\tilde{\mu}_M = 0,13\tilde{\mu}_C(20) + 0,87\tilde{\mu}_E = 0,034 .$$

5.8 At 31 December 2008 the yield on long-term South African index-linked gilts was 3,7% a year. If, on the expectations hypothesis of the term structure of interest rates (e.g. Cox, Ingersoll & Ross, 1981), this is taken as indicative of currently expected future short-term real interest rates then we can take the ex-ante mean risk-free rate as:

$$\tilde{\mu}_I = 0,037;$$

so that the ex-ante expected return on the market portfolio is:

$$\begin{aligned} \hat{\mu}_M &= 0,037 + 0,034 \\ &= 0,071. \end{aligned}$$

5.9 On the basis of the exponential AR model the ex-post estimate of  $\mu_M$  is:

$$\frac{1}{21} \sum_{t=2}^{22} \left[ \delta_{it} + 0,01 \exp \left\{ -13,72 \left( \delta_{M,t-1} - \delta_{I,t-1} \right) \right\} \right] = 0,079 .$$

If that model is used for predictive purposes, this means that, for ex-ante estimation, the expected value of  $\delta_{Mt}$  must be reduced from 0,079 to 0,071. As parameterised, the formula of the exponential AR model is:

$$\delta_{Mt} = \delta_{Mt-1} + 0,01 \exp\{-13,72(\delta_{Mt-1} - \delta_{t,t-1})\} + \sigma_M \varepsilon_t.$$

In order to reduce the estimate of  $\mu_M$  to 0,071,  $h$  could be reduced below 0,01, or  $\delta_{Mt}$  may be multiplied by a constant less than unity, or a constant may be deducted from the right-hand side of the latter equation. If  $h$  is reduced below 0,01, the problem of the relaxation of the required constraints discussed in ¶4.5.4 is exacerbated. If  $\delta_{Mt}$  is multiplied by a constant less than unity, or if a constant is deducted from the right-hand side, the basic constraints are violated, and there is a non-zero probability that, conditionally on information at the start of year  $t$ , the expected excess return on the market during that year will be negative.

5.10 In the light of the problems discussed in section 4.5 and the further difficulties arising here, it must be accepted that the exponential AR model cannot generally be used for predictive purposes. The basic model must therefore be used. For that purpose consideration must be given to the adjustment of the value of  $g$ .

5.11 On the basis of the basic model, the ex-post estimate of  $\mu_M$  is:

$$1,57\bar{\delta}_t = 0,070.$$

As it happens, this is very close to the ex-ante value of 0,071. Nevertheless, in order to increase  $\mu_M$  to 0,071, we increase  $g$  to:

$$1,76 \frac{0,071}{0,070} = 1,79.$$

Since this satisfies the required constraints and falls within the confidence limits of the original estimate, the latter value is adopted with no further reconsideration of the model.

## 6. SUMMARY AND CONCLUSION

6.1 In this paper, the development of descriptive models of the South African market portfolio is described. The models have the attributes specified in section 1. For the purpose of predictive modelling the best of those models was found to be the basic model. For this purpose, as explained in ¶1.5, it was decided to use ex-ante estimates of expected returns.

6.2 The model is defined as:

$$\delta_{Mt} = g\delta_{Mt-1} + \sigma_M \varepsilon_t;$$

where:

$$g = 1,79;$$

$$\sigma_M = 0,171;$$

and  $\varepsilon_t \sim N(0,1)$  is serially independent

6.3 The model is suitable for use with the equilibrium model developed in Thomson & Gott (unpublished), which includes a model for  $\delta_{It}$  conditional on information at time  $t-1$ , and, as contemplated in ¶1.2, in the pricing of the liabilities in an incomplete market as proposed by Thomson (2005). The model may equally be used to model the South African market portfolio in conjunction with any other model of  $\delta_{It}$ .

6.4 Because the process of ex-ante estimation of the parameter  $g$  involved a subjective assumption regarding  $\tilde{\mu}_E$ , the excess return on equities over the risk-free rate, there is room for professional judgement. However, the process of deriving that parameter may be followed by the user for other assumptions regarding  $\tilde{\mu}_E$ . As time goes by it will be necessary to revisit the parameterisation of the model. In the long run it will also be necessary to revisit the selection of the model. Again, the process used in this paper may be followed. The frequency with which the model is reparameterised and its selection revisited is left to the discretion of the user.

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## APPENDIX

### LIKELIHOOD FUNCTIONS AND MAXIMUM-LIKELIHOOD ESTIMATES OF THE MARKET-PORTFOLIO MODELS

#### A.1 THE BASIC MODEL

A.1.1 The log-likelihood function of the basic model follows that of linear regression (e.g. Hocking, 1996: 136–45), viz.:

$$l(\theta) = -\frac{N}{2} \ln(2\pi\sigma_M^2) - \frac{1}{2\sigma_M^2} \sum_{t=1}^N z_t^2;$$

where:

$$\theta = \begin{pmatrix} g \\ h \end{pmatrix};$$

$$z_t = \delta_{Mt} - g\delta_{It} - h.$$

A.1.2 The maximum-likelihood estimates may be derived from the above equations to give:

$$\hat{g} = \frac{S_{IM}}{S_{II}}; \text{ and}$$

$$\hat{h} = \bar{\delta}_M - \hat{g}\bar{\delta}_I;$$

where:

$$S_{IM} = \sum_{t=1}^N (\delta_{It} - \bar{\delta}_I)(\delta_{Mt} - \bar{\delta}_M); \text{ and}$$

$$S_{II} = \sum_{t=1}^N (\delta_{It} - \bar{\delta}_I)^2.$$

Also, after adjusting for bias:

$$\hat{\sigma}_M^2 = \frac{1}{N-2} \left( S_{MM} - \frac{S_{IM}^2}{S_{II}} \right);$$

where:

$$S_{MM} = \sum_{t=1}^N (\delta_{Mt} - \bar{\delta}_M)^2.$$

#### A.2 THE REGIME-SWITCHING MODEL

Following Hamilton (1989), the log-likelihood function of the regime-switching model is:

$$l(\theta) = \ln f(\delta_{M1} | \theta) + \sum_{t=2}^N \ln f(\delta_{Mt} | \delta_{M,t-1}, \dots, \delta_{M1}, \theta);$$

where:

$$\theta = \begin{pmatrix} p_{00} \\ p_{10} \\ g_0 \\ h_0 \\ \sigma_0 \\ g_1 \\ h_1 \\ \sigma_1 \end{pmatrix};$$

$$f(\delta_{M_t} | \delta_{M,t-1}, \dots, \delta_{M_1}, \theta) = f(\delta_{M_t}, S_t = 0 | \delta_{M,t-1}, \dots, \delta_{M_1}, \theta) + f(\delta_{M_t}, S_t = 1 | \delta_{M,t-1}, \dots, \delta_{M_1}, \theta) \text{ for } t > 1$$

$$f(\delta_{M_1}, S_1 = 0 | \theta) + f(\delta_{M_1}, S_1 = 1 | \theta) \text{ for } t = 1;$$

$$f(\delta_{M_t}, S_t = s | \delta_{M,t-1}, \dots, \delta_{M_1}, \theta) = P\{S_t = s | \delta_{M,t-1}, \dots, \delta_{M_1}, \theta\} f_s(\delta_{M_t} | \theta) \text{ for } t > 1;$$

$$\pi_s f_s(\delta_{M_t} | \theta) \text{ for } t = 1;$$

$$P\{S_t = s | \delta_{M,t-1}, \dots, \delta_{M_1}, \theta\} = p_{0s} P\{S_{t-1} = 0 | \delta_{M,t-1}, \dots, \delta_{M_1}, \theta\} + p_{1s} P\{S_{t-1} = 1 | \delta_{M,t-1}, \dots, \delta_{M_1}, \theta\};$$

$$\pi_0 = \frac{p_{10}}{p_{10} + p_{01}};$$

$$\pi_1 = \frac{p_{01}}{p_{10} + p_{01}}; \text{ and}$$

$$f_s(\delta_{M_t} | \theta) = \frac{1}{\sqrt{2\pi}\sigma_s} \exp\left\{-\frac{1}{2\sigma_s^2}(\delta_{M_t} - g_s \delta_{M,t-1} - h_s)^2\right\}.$$

**A.3 THE EXPONENTIAL AR MODEL**

The log-likelihood function of the exponential AR model is:

$$l(\theta) = -\frac{1}{2}\{(N-1)\ln(2\pi\hat{\sigma}^2) + N-3\};$$

where:

$$\theta = \begin{pmatrix} g \\ h \\ \alpha \end{pmatrix};$$

$$\hat{\sigma}^2 = \frac{1}{N-3} \sum_{t=2}^N (z_t - h e^{\alpha z_{t-1}})^2; \text{ and}$$

$$z_t = \delta_{M,t} - g \delta_{I,t}.$$

As explained in section 4.3, the log-likelihood is adjusted to:

$$\tilde{l}(\theta) = \frac{N}{N-1} l(\theta).$$

#### A.4 THE ARCH MODEL

For the ARCH model the log-likelihood function for the estimation of  $a$  and  $b$  is:

$$l(\theta) = -\frac{N}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=2}^N \left\{ \ln(\sigma_t^2) + \left( \frac{z_t}{\sigma_t} \right)^2 \right\};$$

where:

$$\theta = \begin{pmatrix} a \\ b \end{pmatrix};$$

$$z_t = \delta_{M,t} - g \delta_{I,t} - h; \text{ and}$$

$$\sigma_t^2 = a + b z_{t-1}^2.$$