

Actuarial Society of South Africa

WRITTEN EXAMINATION SOLUTIONS

APRIL 2019

Subject A213 — Contingencies

QUESTION 1

$$p_{65} = 0.9706$$

$$p_{66} = 0.9641$$

i.

$${}_{0.8}p_{65.5} = {}_{0.5}p_{65.5} * {}_{0.3}p_{66}$$

$$\begin{aligned} {}_{0.5}q_{65.5} &= 0.5 * q_{65} / (1 - 0.5 * q_{65}) \\ &= 0.0147 / (1 - 0.0147) = 0.0149 \end{aligned}$$

$$\begin{aligned} {}_{0.3}q_{66} &= 0.3 * q_{66} \\ &= 0.3 * 0.0359 = 0.0108 \end{aligned}$$

$$\text{Hence } {}_{0.8}p_{65.5} = (1 - 0.0149) * (1 - 0.0108) = 0.9745$$

ii.

$$\mu_t = -\ln(p_t)$$

$$\mu_{65} = -\ln(0.9706) = 0.0298$$

$$\mu_{66} = -\ln(0.9641) = 0.0366$$

Hence:

$$\begin{aligned} {}_{0.8}p_{65.5} &= {}_{0.5}p_{65.5} * {}_{0.3}p_{66} \\ &= e^{-0.5 * 0.0298} * e^{-0.3 * 0.0366} \\ &= 0.9852 * 0.9891 \\ &= 0.9744 \end{aligned}$$

Examiner comments:

Students did generally well on this question which was pleasing. Alternative approaches given the same results scored full credit. Some candidates have made some basic errors in calculating the life table and hence lost out valuable marks here.

QUESTION 2

i.

EPV

$$\begin{aligned} &= 500,000A_{40} - v^{12} \cdot {}_{12}p_{40} \cdot 100,000 A_{52} - v^{17} \cdot {}_{17}p_{40} \cdot 100,000 A_{57} - v^{20} \cdot {}_{20}p_{40} \cdot 300,000 A_{60} \\ &= 500,000 * 0.23056 \\ &- 100,000 \cdot v^{12} \frac{l_{52}}{l_{40}} \cdot (0.35238) - 100,000 \cdot v^{17} \frac{l_{57}}{l_{40}} \cdot (0.41556) - 300,000 \cdot v^{20} \frac{l_{60}}{l_{40}} \cdot (0.45640) \quad \color{red}{\downarrow} \\ &= 500,000 * 0.23056 - 100,000 * 0.21572 - 100,000 * 0.20492 - 300,000 * 0.19627 \\ &= 14,335.26 \end{aligned}$$

ii.

EPV

$$\begin{aligned} &= 75,000 (A_{\overline{57}|3} - v^3 \cdot {}_3p_{57}) \\ &= 75,000 * (0.84036 - (1.06)^{-3} \cdot \frac{l_{60}}{l_{57}}) \\ &= 75,000 * (0.84036 - 0.825017) \\ &= 75,000 * 0.015343 \\ &= 1150.72 \end{aligned}$$

Examiner comments:

Students scored well on question i, but many misread the question in item ii) and interpreted R425k as the new benefit level which was incorrect. Some missed the point around select mortality.

QUESTION 3

- i. The dependent probability ${}_t(aq)_x^\alpha$ is the probability that a life aged x in a particular state will be removed from that state within t years by decrement α , in the presence of all other decrements in the population.
- ii. The independent probability ${}_tq_x^\alpha$ is the probability that a life aged x in a particular state will be removed from that state within t years by the decrement α , where α is the only decrement acting on the population.
- iii. We know that ${}_tP_x^{\overline{fj}} = \exp\left[-\int_0^t \sum_{j \neq f} \mu_{x+s}^{fj} ds\right]$

From the Kolmogorov Forward Equations and assumed knowledge but can be relatively easily derived from the equations in the tables on page 33

Now since $\rho_x = 0$ and $\nu_x = 0$ return to the active or healthy state is impossible, so

$${}_t(ap)_x = {}_tP_x^{HH} = {}_tP_x^{\overline{HH}}$$

Assuming constant transition intensities:

$${}_t(ap)_x = {}_tP_x^{\overline{HH}} = \exp\left[-\int_{s=0}^t (\mu_{x+s}^{HF} + \mu_{x+s}^{HD}) ds\right] = \exp\left[-\int_{s=0}^t (\sigma_{x+s} + \mu_{x+s}) ds\right]$$

$${}_t(ap)_x = e^{-(\mu+\sigma)t}$$

And for the other probabilities again assuming constant transition intensities:

$$\frac{\partial}{\partial t} {}_t(aq)_x^f = \sigma {}_t(ap)_x = \sigma e^{-(\sigma+\mu)t}$$

$$\frac{\partial}{\partial t} {}_t(aq)_x^d = \mu {}_t(ap)_x = \mu e^{-(\sigma+\mu)t}$$

With closed form solutions with $t=1$, obtained by integrating the differential equations from 0 to 1:

$$\left[{}_t(aq)_x^f\right]_0^1 = \int_0^1 \sigma e^{-(\sigma+\mu)t} dt$$

$$= (aq)_x^f - {}_0(aq)_x^f = -\frac{\sigma}{\mu + \sigma} \left[e^{-(\mu+\sigma)t} \right]_0^1$$

$$\Rightarrow (aq)_x^f = \frac{\sigma}{\sigma + \mu} \left[1 - e^{-(\mu+\sigma)} \right]$$

$$as \quad {}_0(aq)_x^f = 0$$

And now with $t \neq 1$ and assuming the transition intensities are constants over $[x; x+t]$ we have:

$${}_t(aq)_x^f = \frac{\sigma}{\sigma + \mu} \left[1 - e^{-t(\mu+\sigma)} \right]$$

Examiner comments:

A straight forward bookwork question in which student scored particularly poorly.

QUESTION 4

The first result is proved as follows:

$${}_n|A_x = E(J) \quad \text{where:} \quad J = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$$

Now $A_x = E(X)$, say, where:

$$X = v^{K_x+1}$$

and $A_{x:n}^1 = E(Y)$, say, where:

$$Y = \begin{cases} v^{K_x+1} & \text{if } K_x < n \\ 0 & \text{if } K_x \geq n \end{cases}$$

So:

$$\begin{aligned} X - Y &= \begin{cases} v^{K_x+1} - v^{K_x+1} & = 0 & \text{if } K_x < n \\ v^{K_x+1} - 0 & = v^{K_x+1} & \text{if } K_x \geq n \end{cases} \\ &= J \end{aligned}$$

Therefore:

$$E(J) = E(X - Y) = E(X) - E(Y) = A_x - A_{x:n}^1$$

Furthermore:

$$\begin{aligned}
 A_x - A_{x:\overline{n}|}^1 &= \sum_{k=0}^{\infty} v^{k+1} P(K_x = k) - \sum_{k=0}^{n-1} v^{k+1} P(K_x = k) \\
 &= \sum_{k=n}^{\infty} v^{k+1} P(K_x = k) \\
 &= \sum_{k=n}^{\infty} v^{k+1} {}_k p_x q_{x+k} \\
 &= v^n {}_n p_x \sum_{k=n}^{\infty} v^{k+1-n} {}_{k-n} p_{x+n} q_{x+k}
 \end{aligned}$$

since ${}_k p_x = {}_n p_x {}_{k-n} p_{x+n}$ (for $k \geq n$).

If we let $j = k - n$ in the summation, we can write:

$$\begin{aligned}
 A_x - A_{x:\overline{n}|}^1 &= v^n {}_n p_x \sum_{j=0}^{\infty} v^{j+1} {}_j p_{x+n} q_{x+n+j} \\
 &= v^n {}_n p_x \sum_{j=0}^{\infty} v^{j+1} P(K_{x+n} = j) \\
 &= v^n {}_n p_x A_{x+n}
 \end{aligned}$$

as required.

Examiner comments:

The questions specifically asked for first defining the random variable and first principles being implied that the approach should follow on from the random variable being defined. Some students attempt to “backsolve” the proof i.e. working from the top and then bottom and then the middle step has some steps missing which resulted in poor marks being scored.

QUESTION 5

i.

The compound bonuses vest before the payment of the death benefit. The expected value of the benefit is

$$EPV = 1\,000\,000 \left(1.01923 v_{\overline{d}_{35}} + (1.01923)^2 v_{\overline{2d}_{36}} + \dots \right)$$

$$\text{Let } v^* = 1.01923v = \frac{1.01923}{1.06} = \frac{1}{1.04}$$

We can now value the benefit as an endowment assurance using an interest rate of 4%

$$EPV = 1\,000\,000 A_{35:20} \dots \dots \dots (1)$$

Let P denote the monthly premium

$$EPV(\text{Premiums}) = 12P a_{35:20}^{(\ddot{i}2)} \dots \dots \dots (2)$$

The expected present value of the expenses is

$$EPV(\text{Expenses}) = (0.45)(12P) + (0.03)(12P)(a_{35:20}^{(12)} - \ddot{a}_{35:1}^{(12)})$$

From (1) above

$$\begin{aligned} A_{35:20} &= A_{35} - v_{\overline{20} \frac{l_{55}}{l_{35}}} A_{55} + v_{\overline{20} \frac{l_{55}}{l_{35}}} \\ &= (0.19219) - (1.04)^{-20} * \frac{9\,557.8179}{9\,894.4299} * (0.38950) + (1.04)^{-20} * \frac{9\,557.8179}{9\,894.4299} \\ &= 0.461335 \end{aligned}$$

$$\begin{aligned} a_{35:20}^{(\ddot{i}2)} &= \left(\ddot{a}_{35} - \frac{11}{24} \right) - v_{6\%}^{20} * \frac{l_{55}}{l_{35}} * \left(a_{55} - \frac{11}{24} \right) \\ &= \left(15.990 - \frac{11}{24} \right) - (1.06)^{-20} * \frac{9\,557.8179}{9\,884.4299} * \left(13.057 - \frac{11}{24} \right) \\ &= 11.73669 \end{aligned}$$

$$\begin{aligned} a_{35:1}^{(\ddot{i}2)} &= \left(\ddot{a}_{35} - \frac{11}{24} \right) - v_{6\%} * \frac{l_{36}}{l_{35}} * \left(a_{36} - \frac{11}{24} \right) \\ &= \left(15.990 - \frac{11}{24} \right) - (1.06)^{-1} * \frac{9\,887.6126}{9\,884.4299} * \left(15.901 - \frac{11}{24} \right) \\ &= 0.973151 \end{aligned}$$

Solving equation of value

$$(12P)(11.73669) = 1\,000\,000 \cdot (0.461335) + 5.4P + 0.36P(11.73669 - 0.973151)$$

$$P = \frac{1\,000\,000 \cdot (0.461335)}{(12 \cdot 11.73669) - 5.4 - (0.36 \cdot 10.76383)}$$

$$\underline{P = 3\,506.42}$$

ii.

The expected present value of the future benefits at the start of the 16th policy year is

$$EPV = 1\,000\,000 \left[(1.01923)^{16} v_{l_{50}}^{d_{50}} + (1.01923)^{17} v_{l_{50}}^{2d_{51}} + \dots \right]$$

$$EPV = 1\,000\,000 \cdot (1.01923)^{15} \left[(1.01923) v_{l_{50}}^{d_{50}} + (1.01923)^2 v_{l_{50}}^{2d_{51}} + \dots \right]$$

$$EPV = 1\,000\,000 \cdot (1.01923)^{15} \left(v_* \frac{d_{50}}{l_{50}} + v_*^2 \frac{d_{51}}{l_{50}} + \dots \right)$$

$$EPV = 1\,000\,000 \cdot (1.01923)^{15} A_{50:5}^{4\%}$$

$$\text{Where } v_* = \frac{1.01923}{1.06} = \frac{1}{1.04} \text{ as before}$$

The expected present value of expenses at the start of the 16th policy year is

$$EPV(\text{Expenses}) = (0.03)(12P) a_{50:5}^{(12)}$$

The expected present value of the future premiums at start of 16th policy year is

$$EPV(\text{Premium}) = 12P a_{50:5}^{(12)}$$

$$A_{50:5}^{4\%} = A_{50} - v_{l_{50}}^{5l_{55}} A_{55} + v_{l_{50}}^{5l_{55}}$$

$$= (0.32907) - (1.04)^{-5} * \frac{9\,557.8179}{9\,712.0728} * (0.38950) + (1.04)^{-5} * \frac{9\,557.9179}{9\,712.0728}$$

$$= 0.822887$$

$$a_{50:5}^{(12)} = \left(a_{50} - \frac{11}{24} \right) - v_*^5 * \frac{l_{55}}{l_{50}} * \left(a_{55} - \frac{11}{24} \right) \text{ at } 6\%$$

$$= \left(14.044 - \frac{11}{24} \right) - (1.06)^{-20} * \frac{9\,557.8179}{9\,712.0728} * \left(13.057 - \frac{11}{24} \right)$$

$$= 4.320738$$

The expected present value of the future benefit

$$EPV = 1\,000\,000 \cdot (1.01923)^{15} \cdot 0.822887 + (0.03)(12 \cdot 3\,500)(4.320738) - (12 \cdot 3\,500 \cdot 4.320738)$$

$$\underline{EPV = 918\,995.60}$$

iii.

- The expected cost of paying benefits increases over the term of the contract as the probability of death/maturity increases with age, whereas the premiums stay level.
- This means that the premiums received in earlier years of a contract exceed the benefits that are due in those years, but in later years, the premiums are too small to pay for the benefits.
- it is therefore prudent (or a regulatory requirement) for the premiums that are required in the early years of a contract to be set aside to fund the shortfall in the later years of the contract,

Examiner comments:

Most students were able to make a decent attempt at this question. Many missed out on marks for using age 51 and not 50 in part ii.

QUESTION 6

The per policy reserves at the end of the first year are:

The per policy reserves at the end of the first year are:

$$V_1 = 1\,500\,000 * (1.04)^{0.5} * (A_{51-v^{14}} p_{51} A_{65}) \quad @4\%$$

$$V_1 = 1\,500\,000 * (1.04)^{0.5} * (A_{48-v^{14}} p_{48} A_{62}) \quad @ 4\%$$

$$V_1 = 1\,500\,000 * (1.04)^{0.5} * (0.30695 - (1.04)^{-14} * \frac{9\,129.7170}{9\,753.4714} * (0.48458))$$

$$V_1 = 68\,856.66$$

$$DSAR = S * (1+i)^{0.5} - V_1$$

$$DSAR = 1\,500\,000 * (1.04)^{0.5} - 68\,856.66$$

$$DSAR = 1\,460\,849.19$$

$$EDS = 20\,000 * q_{47} * DSAR$$

$$EDS = 20\,000 * 0.001802 * 1\,460\,849.19$$

$$EDS = 52\,649\,004.95$$

$$ADS = 20 * 1\,460\,849.19$$

$$ADS = 29\,216\,983.88$$

$$\text{Mortality Profit} = EDS - ADS$$

$$\text{Mortality Profit} = 52\,649\,004.95 - 29\,216\,983.88$$

$$\text{Mortality Profit} = \underline{23\,432\,021.07}$$

Examiner comments:

Many students wasted time by following a retrospective reserve calculation.

QUESTION 7

Let x be male (Peter) and y be female (Sandy).

i.

Let P be the monthly premium. Then equation of value:

$$12P\ddot{a}_{52:50:\overline{5}|}^{(12)} = 30000 + 500000\bar{A}_{52:50}$$

With

$$\ddot{a}_{52:50:\overline{5}|}^{(12)} = \ddot{a}_{52:50}^{(12)} - v^5 \frac{l_{57:55}}{l_{52:50}} \ddot{a}_{57:55}^{(12)} \dots (1)$$

$$\ddot{a}_{52:50} = 17.295$$

$$\ddot{a}_{57:55} = 15.558$$

$$(1) = \ddot{a}_{52:50} - \frac{11}{24} - v^5 \frac{l_{57:55}}{l_{52:50}} \left[\ddot{a}_{57:55} - \frac{11}{24} \right]$$

$$\frac{l_{57:55}}{l_{52:50}} = \frac{9880.196 \times 9917.623}{9930.244 \times 9952.697} = 0.991453734$$

$$(1) = 4.531907545$$

Furthermore,

$$A_{52:50} = 1 - d\ddot{a}_{52:50} = 1 - \frac{0.04}{1.04} [17.295] = 0.3348077$$

$$\bar{A}_{52:50} = 1.04^{0.5} A_{52:50} = 0.341438199$$

Bringing it all together:

$$12P[4.531907545] = 30000 + 500000[0.341438199]$$

$$\therefore P = R3690.85$$

ii.

Events leading to premiums being stopped:

- The death of either or both of the lives
- The end of the premium paying period
- Policy cancellation or surrender
- The policy being made paid up
- The benefit is being paid
- Death of the premium payer
- When it becomes impossible for the benefit to be paid in the future

iii.

The likely impact on premium:

- Premiums payable monthly in advanced can be approximated as being paid annually half way through the year
- Premiums paid annually in arrears are paid at the end of the year, effectively on average 6 months later than monthly in advance
- This means that premiums would earn on average 6 month's less interest at 4% when paid annually in arrears
- Meaning that if the company want to be profit neutral, they would have to increase the premium rate
- There is also a higher risk that either Peter or Sandy dies during a year meaning that the insurer has increased mortality risk, further requiring an increase in premiums to be charged

Examiner comments:

A standard question done well by well-prepared students.

QUESTION 8

$$EPV = 20000 \times 12 \times \ddot{a}_{60:60:\overline{25}|}^{(12)}$$
$$= 20000 \times 12 \times \left[\ddot{a}_{60:25|}^{(12)M} + \ddot{a}_{60:25|}^{(12)F} - \ddot{a}_{60:60:\overline{25}|}^{(12)} \right]$$

now

$$\ddot{a}_{60:25|}^{(12)M} = \ddot{a}_{60}^{(12)M} - v^{25} {}_{25}P_{60} \left[\ddot{a}_{85}^M - \frac{11}{24} \right]$$
$$= 15.632 - \frac{11}{24} - 1.04^{-25} \frac{4892.878}{9826.131} \left[5.842 - \frac{11}{24} \right]$$
$$= 14.168063$$

and

$$\ddot{a}_{60:25|}^{(12)F} = \ddot{a}_{60}^{(12)F} - v^{25} {}_{25}P_{60} \left[\ddot{a}_{85}^F - \frac{11}{24} \right]$$
$$= 16.652 - \frac{11}{24} - v^{25} \frac{6122.154}{9848.431} \left[7.22 - \frac{11}{24} \right]$$
$$= 14.616936$$

also

$$\ddot{a}_{60:60:\overline{25}|}^{(12)} = \ddot{a}_{60:60}^{(12)} - v^{25} {}_{25}P_{60:60} \ddot{a}_{85:85}^{(12)}$$
$$\approx \ddot{a}_{60:60} - \frac{11}{24} - v^{25} {}_{25}P_{60:60} \left[\ddot{a}_{85:85} - \frac{11}{24} \right]$$
$$= 14.090 - \frac{11}{24} - v^{25} \frac{4892.878}{9826.131} \frac{6122.154}{9848.431} \left(4.396 - \frac{11}{24} \right)$$
$$= 13.1744$$

hence

$$EPV = 240000(14.168063 + 14.616936 - 13.1744)$$
$$= R3\ 746\ 543.76$$

Examiner comments:

A standard question done well by well-prepared students.