Actuarial Society of South Africa

EXAMINERS REPORTS

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Subject A211

General comments

Please note that different answers may be obtained to those shown in these solutions depending on whether intermediary figures were obtained from tables or from calculators, but candidates were not penalised for this.

Also, note that there are often alternative ways to reach the same final solution so that the solutions in this report should not be seen as the only solutions available.

Many candidates can also increase their probability of passing this exam by practising good exam technique, which is often missing in its most basic form.

Question 1

- The researchers have defined a set of objectives "to provide a model to accurately predict the growth of the for South Africa population. "
- For this to be met, they need to plan the modelling process by including all factors, for example differentiating between sexes, or region, or migration etc.
- This requires data collection and analysis to make sure the model captures the real South African population growth.
- Most developing countries like South Africa are affected by lack of accurate records of birth , death and migration. This is because most births and deaths are not documented because they happen outside hospital systems. Migration is also affected due to undocumented foreigners who visit and leave without trace.
- Define the parameters for the model and consider appropriate values e.g., values for birth rates and mortality rates.
- Capturing the essence of the real-world system, e.g., the interaction between emigration and immigration.
- Involve experts on the real world system like experts in population growth, economists etc.
- Simulation package or a general-purpose e.g., buy a software package that can predict population growth.?
- Write the computer program for your model e.g., use R-code to write your own program for predicting population growth.
- Debug the program to make sure the program performs the intended operations.
- Test the reasonableness of the output. Does the population growth appear reasonable in the short, medium and long term.
- Review and carefully consider the appropriateness in the light of small changes in input parameters e.g. doe sensitivity analysis.
- Analyse the output from the model. How does output compare to previous models of population growth?
- Ensure that any relevant professional guidance has been considered, for instance any professional guidance from ASSA.
- Communicate and document the results and the model. Write a report or prepare a .pdf presentation for a presentation at a conference.

[Total 9]

Question was done well.

The common error was to not give an example at each step.

Question 2

- Initial +'ve cashflow (loan from the bank): size/timing certain
- Initial -'ve cashflow (buying taxi): size/timing certain.
- When the taxi operates, the revenue (+'ve income):
  Amount uncertain - not known with certainty since the number of passengers is uncertain.
  Timing certain/uncertain *(can be either one but needs to justify)* – daily every time, every time taxi operates.
- Operational cost (-ve cashflow) – upkeep of taxi/diesel/salaries/insurance – size/timing uncertain
- Loan repayments (-ve cashflow) – depending on loan contract.
  -timing certain
  -amounts certain/uncertain - depending on interest rate of loan (interest rate can be fixed or variable) *(can be either one but needs to justify)*
- Final cashflow - +ve (sale of taxi) size/timing
  uncertain

[Total 4]

Common errors were:

- *Not mentioning all the necessary cashflow.*
- *Leaving out either the timing or size of the cashflow, although the question clearly asked for both.*

The justification asked for in the model solution is one word justification, e.g., variable or fixed interest rate.

Question 3

$$PV = 200e^{-0.05} + 100e^{-3\times0.05} + e^{-5\times0.05}\int_5^{10}(50-6t)\exp[-\int_5^t(0.05-0.006s)ds]dt$$

$$= 190.246 + 86.0708 + e^{-5\times0.05}\int_5^{10}(50-6t)\exp[-\int_5^t(0.05-0.006s)ds]dt$$

$$= 276.317 + e^{-0.25}\int_5^{10}(50-6t)\exp[-(0.05s-0.003s^2)|_5^t]dt$$

$$= 276.317 + e^{(-0.25+0.25-0.075)}\int_5^{10}(50-6t)\exp[-(0.05t-0.003t^2)]dt$$

$$= 276.317 + e^{-0.075}\int_{0.175}^{0.2}e^{-u}1{,}000du$$

where we make the substitution u = $0.05t-0.003t^2$, new limits of the integral

so that $du=(0.05-0.006t)dt$ and $1000du=(50-6t)dt$ (** see *Alternative A*)

$$= 276.317 + e^{-0.075}\times1{,}000\left[e^{-u}\right]_{0.175}^{0.2}$$

$$PV = 276.317 - 1000\,e^{-0.075}\left(e^{-0.2}-e^{-0.175}\right)$$

$$= 276.317 + 19.2286 = 295.546 = R295.55$$

(**) *Alternative A*

$$\frac{d}{dt}\left(0.05t-0.003t^2\right) = 0.05-0.006t$$

$$\int_5^{10}(50-6t)\exp[-(0.05t-0.003t^2)]dt = 1{,}000\int_5^{10}(0.05-0.006t)\exp[-(0.05t-0.003t^2)]dt$$

$$= 1{,}000\left[\exp[-(0.05t-0.003t^2)\right]_5^{10}$$

Integral of the form $\int f'(t)e^{f(t)}dt = e^{f(t)}$

[Total 11]

This type of question appears in each exam session and in general is always answered well.

It should be remembered that a numerical answer without showing exactly how the integration was done will not get full credit even if the numerical answer is correct.

Common errors:

- *Errors in the boundary conditions of the 2nd integral of the 3rd term.*
- *Not showing how the integration was done.*

Question 4

i.

Given that $P_t = \dfrac{\beta(\beta+1)}{(\beta+t)(\beta+t+1)}$

We know that $Y_t = -\dfrac{1}{t}\ln(P_t) = -\dfrac{1}{t}\ln\left(\dfrac{\beta(\beta+1)}{(\beta+t)(\beta+t+1)}\right)$

$= -\dfrac{1}{t}\left(\ln\beta(\beta+1) - \ln(\beta+t) - \ln(\beta+t+1)\right)$

[2]

ii.

Instantaneous forward rate $F_t = -\dfrac{d}{dt}\ln P_t$ or $F_t = -\dfrac{1}{P_t}\dfrac{d}{dt}P_t$.

Now, from (i) $\ln P_t = \ln\left(\dfrac{\beta(\beta+1)}{(\beta+t)(\beta+t+1)}\right) = \ln\left(\beta(\beta+1)\right) - \ln(\beta+t) - \ln(\beta+t+1)$

$\Rightarrow -\dfrac{d}{dt}\ln P_t = \dfrac{1}{\beta+t} + \dfrac{1}{\beta+t+1} = \dfrac{2\beta+2t+1}{(\beta+t)(\beta+t+1)}$

Therefore $F_5 = \dfrac{1}{5+\beta} + \dfrac{1}{6+\beta} = \dfrac{2\beta+11}{(\beta+5)(\beta+6)}$

[5]
[Total 7]

This question was not done well.

Common error is unfamiliarity with the underlying theory.

Question 5

i.

$PV_L = 300,000v^{20} + 150,000a^{(2)}_{\overline{30}|i}v^{15}$ @ $i = 0.065$

and $a^{(2)}_{\overline{30}|0.065} = 13.2675$ and $v = 0.938967$

$PV_L(0.065) = 85,139.108676426 + 773,815.649782275 = 858,954.7584587007$.

Let B be the nominal amount for Bond B at redemption in n years' time. Then for assets we have $PV_{As}(i) = 450,000v^{10} + Bv^n$

$858,954.75845870077564 = 450,000v^{10} + Bv^n$ and

With $450,000v^{10} = 239,726.715984238$

Bv^n = R619,228 is the amount of money invested in Bond B.

[7]

ii.

Given that at $i = 6.5\%$ the numerator of DMT for the liabilities equals 21,704,293.03, then

DMT $_L$ = $\frac{21,704,293.03}{858,954.76} = 25.2683$. Therefore $Vol_L = DMT_L \times v = 23.7261$

Thus $Numerator\ Vol_L = Numerator\ DMT_L \times v$

For the assets, we have $PV_{As}(i) = 450,000v^{10} + Bv^n$ so that

$$-\frac{d}{di}PV_{As}(i) = 10 \times 450,000v^{11} + nBv^{n+1}$$

$$-\frac{d}{di}PV_{As}(i) = 10 \times v \times 450,000v^{10} + n \times v \times Bv^n$$

$$\Rightarrow -\frac{d}{di}PV_{As}(0.065) = 2,250,955.0796642043 + 0.938967 \times 619228n$$

$$\Rightarrow \frac{20,379,741.435636 - 2,250,955.0796642043}{581,435} = n \Rightarrow n = 31.1794 \text{ years}$$

[6]
[Total 13]

Part (i) was the best answered question in the paper.

Question 6

i.
If $i^{(4)} = 0.09$ then $i = 0.09308$.

The equation of value is

100,000 =
$$\left(vY + (1.06)v^2Y + \ldots + (1.06)^{14}v^{15}Y\right) + Y \times (1.06)^{14} \times (1.1) \times v^{16} + \ldots + Y(1.06)^{14} \times (1.1)^{15} \times v^{30}$$

$$100{,}000 = Y \times v\left(1 + (1.06)v^1 + \ldots + (1.06)^{14}v^{14}\right) + Y \times (1.06)^{14} \times v^{15}\left((1.1)v^1 + \ldots + (1.1)^{15}v^{15}\right)$$

Let $w_1 = \left[\frac{1.06}{1.09308}\right]^{-1} - 1 = \left[0.969734\right]^{-1} - 1 = 0.031210678$ and
$w_2 = \left[\frac{1.1}{1.09308}\right] - 1 = 1.00633 - 1 = 0.00632768$.

Then 100,000 = $Y \times v \times \ddot{a}_{\overline{15}|w_1} + Y \times (1.06)^{14} \times v^{15} \times \ddot{s}_{\overline{15}|w_2}$ (** <u>Alternative B</u>)

$\ddot{a}_{\overline{15}|w_1}$ = 12.203413106 and $\ddot{s}_{\overline{15}|w_2}$ = 15.78221906

100,000 = (11.164211270 + 9.38968414) × Y $\Rightarrow Y = R4,865.26$

<u>*(**) Alternative B*</u>

$m_1 = \left[\frac{1.06}{1.09308}\right]$ $\qquad$ $m_2 = \left[\frac{1.1}{1.09308}\right]$

$$vY \times v \times \left(\frac{1 - m_1^{15}}{1 - m_1}\right) + Y \times 1.06^{14} \times (1.1) \times v^{16}\left(\frac{1 - m_2^{15}}{1 - m_2}\right)$$

[10]

ii.

The total amount paid in the 30 repayments.

$Y\left(1+1.06+\ldots+1.06^{14}\right) + 1.06^{14}Y\left(1.1+\ldots+1.1^{15}\right)$

$= \text{Y } s_{\overline{15}|6\%} + \text{Y}\times 1.06^{14}\times \ddot{s}_{\overline{15}|10\%}$ (** *Alternative C*)

$s_{\overline{15}|6\%} = 23.275969885$ $\quad \ddot{s}_{\overline{15}|10\%} = 34.949729864$

$= 4,865.257801888\times\left(23.275969885+79.017982501\right)$

$= 4,865.257801888\times 102.293952386$

$= 497,686.449932427$

Total interest paid $= 497,686.449932427 - 100,000 = R397,686.45$

(**) Alternative C

$Y\left(\dfrac{1-1.06^{15}}{1-1.06}\right) + Y\times 1.06^{14}\left(\dfrac{1.1-1.1^{16}}{1-1.1}\right) = 113,244 + 384,443 = 497,687$

iii.

Flat rate = $F = \dfrac{497,687-100,000}{30\times 100,000}$ = 0.13256215= 13.2562150% per annum

[2]
[Total 18]

Part (i) was one of the best answered questions in the paper. The common error in part(i) was to start the growth of the payments one year to early. Examiners mark with error so a total of 1 mark would have been lost for this error if the student worked consistently with their equation of value.

A hall mark of good exam technique is to always start with an equation of value.

An area of concern is when student's do not start with an equation of value and only write down the final equation from which the answer is then calculated. If the final equation is incorrect there are very little credit available in this question.

In part (ii) the total payments on the loan had to be calculated. The common error in part (ii) was to still discount the payments by using technique developed in part (i).

Question 7

There are 4 coupons payable after purchase and the redemption amount. The table below summarizes the money and real cashflows:

Money amount

Date	Cashflow	Reason	Money values
$1-01-2021$	4	Coupon	$4\times\frac{INDEX_{OCT\,2020}}{INDEX_{JAN\,2020}}=4\times\frac{133.6}{131.2}$
$01-07-2021$	4	Coupon	$4\times\frac{INDEX_{APRIL\,2021}}{INDEX_{JAN\,2020}}=4\times\frac{134.2}{131.2}$
$01-01-2022$	4	Coupon	$4\times\frac{INDEX_{OCT\,2021}}{INDEX_{JAN\,2020}}=4\times\frac{134.9}{131.2}$
$01-07-2022$	104	Coupon+ Redemption	$104\times\frac{INDEX_{APRIL\,2022}}{INDEX_{JAN\,2020}}=104\times\frac{136}{131.2}$

Real cashflow

Date	Cashflow	Reason	Real values
$1-01-2021$	$4\times\frac{133.6}{131.2}$	Coupon	$4\times\frac{133.6}{131.2}\times\frac{INDEX_{JULY\,2020}}{INDEX_{JAN\,2021}}=4\times\frac{133.6}{131.2}\times\frac{133.1}{133.9}$
$01-07-2021$	$4\times\frac{134.2}{131.2}$	Coupon	$4\times\frac{134.2}{131.2}\times\frac{INDEX_{JULY\,2020}}{INDEX_{JULY\,2021}}=4\times\frac{134.2}{131.2}\times\frac{133.1}{134.4}$
$01-01-2022$	$4\times\frac{134.9}{131.2}$	Coupon	$4\times\frac{134.9}{131.2}\times\frac{INDEX_{JULY\,2020}}{INDEX_{JAN\,2022}}=4\times\frac{134.9}{131.2}\times\frac{133.1}{135.2}$
$01-07-2022$	$104\times\frac{134.9}{131.2}$	Coupon+Redemption	$104\times\frac{134.9}{131.2}\times\frac{INDEX_{JULY\,2020}}{INDEX_{JULY\,2022}}=104\times\frac{136}{131.2}\times\frac{133.1}{136.1}$

Therefore, the real equation of value is

$$110 = 4.048835134v^{\frac{1}{2}} + 4.051888248v + 4.048922554v^{1\frac{1}{2}} + 105.4285765v^2$$

where i is the real effective rate and $v=\frac{1}{1+i}$

Using interpolation, let P = 110 , $i_1 = 0.035$ and $i_2 = 0.036$.

Let $PV(i) = 4.04884(1+i)^{-0.5} + 4.05189(1+i)^{-1} + 4.04892(1+i)^{-1.5} + 105.429(1+i)^{-2}$, then

$PV(i_1) = 110.159 > P$ and $PV(i_2) = 109.958 < P$

Therefore $i_3 = i_1 + \left(\frac{PV(i_1)-P}{PV(i_1)-PV(i_2)}\right)\times(i_2 - i_1) = 0.0357907$

$PV(i_3) = 110$. Thus the real yield is i = 3.57907% per annum.

[Total 14]

Question was answered reasonably well.

Many marks were available for showing all working.

The common error was the incorrect dates (and therefor incorrect indexes) used when adding and removing inflation.

Like in question 6 examiner mark with error, so even if the incorrect indexes were used full credit could be gained for the 2nd half of the question.

Question 8

i.

Present value of income :

$i = 5\% \Rightarrow i^{(2)} = 4.9390153\% \qquad \delta = 4.87902\%$

$PV(rent) = 500{,}000\ddot{a}_{\overline{1|}}^{(2)} + \left(500{,}000 + 12{,}500\right)\times v \times \ddot{a}_{\overline{1|}}^{(2)} + \left(500{,}000 + 2\times 12{,}500\right)\times v^2 \times \ddot{a}_{\overline{1|}}^{(2)} + \ldots$
$+\left(500{,}000 + 19\times 12{,}500\right)\times v^{19} \times \ddot{a}_{\overline{1|}}^{(2)}$

$= 500{,}000\ddot{a}_{\overline{20|}}^{(2)} + 12{,}500 \times \ddot{a}_{\overline{1|}}^{(2)} \times \left(v + 2v^2 + \ldots + 19v^{19}\right)$

$= 500{,}000\ddot{a}_{\overline{20|}}^{(2)} + 12{,}500 \times \ddot{a}_{\overline{1|}}^{(2)} \times \left(Ia\right)_{\overline{19|}}$ (** *<u>Alternative D</u>*)

$\ddot{a}_{\overline{20|}}^{(2)} = 12.927643221 \qquad \ddot{a}_{\overline{1|}}^{(2)} = 0.987950036$

$$\left(Ia\right)_{\overline{19|}} = \frac{\ddot{a}_{\overline{19|}} - 19v^{19}}{0.05} = 103.412834387$$

$= 6{,}463{,}821.610292730 + 1{,}277.083.918802840 = 7{,}740{,}905.529095580$

<u>(**) Alternative D</u>

$487{,}500\ddot{a}_{\overline{20|}}^{(2)} + 12{,}500 \times \ddot{a}_{\overline{1|}}^{(2)} \times \left(I\ddot{a}\right)_{\overline{20|}}$

$PV(income) = 7{,}740{,}905.529095580 + 21{,}000{,}000v^{20}$
$= 7{,}740{,}905.529095580 + 7{,}914{,}679.140333010 = 15{,}655{,}584.669428600$

$PV(outgo) = 11{,}000{,}000 + 13{,}000\bar{a}_{\overline{20|}} + 2{,}000\left(I\bar{a}\right)_{\overline{20|}} + 1{,}500{,}000v^{15}$ (** *<u>Alternative E</u>*)

$\bar{a}_{\overline{20|}} = 12.77123223$

$$\left(I\bar{a}\right)_{\overline{20|}} = \frac{\ddot{a}_{\overline{20|}} - 20v^{20}}{\delta} = 113.701835127$$

$= 11{,}000{,}000 + 166{,}026.018984500 + 227{,}403.670253793 + 721{,}525.647136455$
$= 12{,}114{,}955.3363747000$

<u>(**) Alternative E</u>

$13{,}000\bar{a}_{\overline{1|}} \times \ddot{a}_{\overline{20|}} + 2{,}000\bar{a}_{\overline{1|}} \times \left(I\ddot{a}\right)_{\overline{20|}}$

$NPV = R3{,}540{,}629.33$

[20]

ii.

Note that $13{,}000\,\bar{a}_{\overline{19|}} + 1{,}500{,}000v^{15} + 2{,}000\left(I\bar{a}\right)_{\overline{19|}} + 11{,}000{,}000 = 11{,}907{,}300.60$

and that $500{,}000\ddot{a}^{(2)}_{\overline{19|}} + 12{,}500 \times \ddot{a}^{(2)}_{\overline{1|}} \times \left(Ia\right)_{\overline{18|}} = 7{,}452{,}568.56 < 11{,}907{,}300.60$

then balance at $t = 19 < 0 \Rightarrow$ DPP > 19 .

Therefore, given the result in (i) positive NPV at $t = 20$, we have DPP = 20.

[4]

[Total 24]

Part (i) was answered well with many marks available for working consistently from the equation of value written down, whether the initial equation of value was correct or not.

Common errors were:

- *Not incorporating yearly increase and half-yearly payments correctly in the PV(rent)*
- *Calculating annuity values incorrectly.*

Part (ii) was not done by many students as many students seem to have run out of time.

